An Interior Ellipsoid Algorithm for Fixed Points*

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Abstract

We consider the problem of approximating fixed points of non-smooth contractive functions with using of the absolute error criterion.

In [12] we proved that the upper bound on the number of function evaluations to compute ε-approximations is $\mathrm{O}(n^3(\ln\frac{1}{\varepsilon}+\ln\frac{1}{1-q}+\ln n))$ in the worst case, where $0 < q < 1$ is the contraction factor and n is the dimension of the problem. This upper bound is achieved by the circumscribed ellipsoid (CE) algorithm combined with a dimensional deflation process.

In this paper we present an inscribed ellipsoid (IE) algorithm that enjoys $\mathrm{O}(n^2(\ln\frac{1}{\varepsilon}+\ln\frac{1}{1-q}+\ln n))$ bound. Therefore the IE algorithm has almost the same (modulo multiplicative constant) number of function evaluations as the (nonconstructive) centroid method [11]. We conjecture that this bound is the best possible for mildly contractive functions ($q \approx 1$) in moderate dimensional case. Affirmative solution of this conjecture would imply that the IE algorithm and the centroid algorithms are almost optimal in the worst case. In particular they are much faster than the simple iteration method, that requires $\left\lceil\frac{\ln(1/\varepsilon)}{\ln(1/q)}\right\rceil$ function evaluations to solve the problem.

Key words: Fixed points, inscribed ellipsoid algorithm, optimal complexity algorithm.

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1 Introduction

Fixed point computation has been an intensive research area since 1967 when Scarf [9] introduced a simplicial continuation algorithm to approximate fixed points. Several classes of methods have been invented since then, including homotopy continuation, simplical and Newton-type methods. Most of these methods solve the problem in the residual sense, i.e., compute x such that the magnitude of $|f(x) - x|$ is small. In our paper we consider the absolute error criterion and the class of contractive functions.

We let $B^n(0,1)$ be the unit ball in the n-dimensional real space R^n and $C^n = [-1,1]^n$ be the unit cube containing that ball. We consider the class of contractive functions

$$F_n = \{f : B^n(0,1) \to B^n(0,1) \ : \ \|f(x) - f(y)\| \leq q\|x - y\|, \ \forall x, y \in B^n(0,1)\} \quad (1.1)$$

where $0 < q < 1$ is the contractive factor and $\|\cdot\|$ is the l_2-norm. We let

$$\bar{f}(x) = \begin{cases} f(x) & x \in B^n(0,1) \\ f(x/\|x\|) & \text{otherwise,} \end{cases} \qquad \text{for } f \in F_n \text{ and } x \in R^n. \quad (1.2)$$

Then we define

$$\bar{F}_n = \{\bar{f} : R^n \to B^n(0,1)\} \quad (1.3)$$

as the extension of the class F_n to functions defined on R^n. It turns out that each $\bar{f} \in \bar{F}$ has the same contraction factor q and fixed point $\alpha = f(\alpha)$ as the corresponding function $f \in F_n$.

We want to compute an approximate solution to the nonlinear equation

$$x = f(x) \quad (1.4)$$

for $f \in F_n$. The Banach's Fixed Point Theorem says that there exists exactly one solution $x^* \in B^n(0,1)$ of (1.4). For any $f \in F_n$, we want to find an *ε-approximation* x_ε to x^* such that

$$\|x_\varepsilon - x^*\| \leq \varepsilon < 1. \quad (1.5)$$

The simple iteration (SI) algorithm given by

$$x_{i+1} = f(x_i), \quad x_0 = 0 \quad (1.6)$$

requires at most

$$n(\varepsilon, q) = \left\lceil \frac{\ln(1/\varepsilon)}{\ln(1/q)} \right\rceil \quad (1.7)$$

iterations (function evaluations) to compute an ε-approximation, for any function $f \in F_n$.

It is known [7] that the efficiency of the SI algorithm can not be essentially improved whenever the dimension $n \geq n(\varepsilon, q)$. For $n < n(\varepsilon, q)$ there exist methods more efficient than the SI algorithm. In the univariate case ($n = 1$) we developed a hybrid bisection-envelope (BEN) algorithm which is minimizing the number of function evaluations. This minimal number is

$$\mathrm{m}(\varepsilon, q) = \left\lceil \frac{\ln(2/\varepsilon)}{\ln((1+q)/q)} \right\rceil < b(\varepsilon, q), \tag{1.8}$$

where $b(\varepsilon, q) = \lceil \log_2(2/\varepsilon) \rceil$ is the number of function evaluations in the bisection algorithm.

In paper [12] we developed a circumscribed ellipsoid (CE) algorithm, for moderate dimensional problems (n not too large) and mildly contractive functions (q close to 1). The number of function evaluations in the CEA algorithm is

$$\mathrm{O}(n^3(\ln \frac{1}{\varepsilon} + \ln \frac{1}{1-q} + \ln n)) \tag{1.9}$$

in the worst case. This algorithm was implemented and tested to be much more efficient than the SI algorithm for small n and q close to 1. Therefore, the CE algorithm is very efficient for highly nonlinear, nonsmooth functions which are almost non-contracting, i.e., for difficult problems.

In this paper we improve the bound from [12]. Namely, we present an inscribed ellipsoid (IE) algorithm, and prove that in the worst case the number of function evaluations is

$$\mathrm{O}(n^2(\ln \frac{1}{\varepsilon} + \ln \frac{1}{1-q} + \ln n)). \tag{1.10}$$

Therefore, the complexity of the IE algorithm is essentially the same as of the (nonconstructive) centroid algorithm [11]. We conjecture that this bound is the best possible in the worst case.

This paper is organized as follows. In section 2 we present preliminary results from our previous work [11, 12] and general results of convex analysis, which are needed in the design and analysis of the IE algorithm. In section 3 we describe the IE algorithm. In section 4, we present the complexity analysis, list some open problems and formulate the conjecture that $\mathrm{O}(n^2(\ln \frac{1}{\varepsilon} + \ln \frac{1}{1-q} + \ln n))$ bound is the best possible.

2 Premilinaries

In the inscribed ellipsoid algorithm presented in this paper, we employ several results and techniques presented in [4, 11, 12, 13]. In particular, a bisection envelope algorithm, dimensional reduction scheme, volume reduction estimates, a fixed point bounding lemma and the Löwner-John ellipsoid theorem are utilized in the design and in the complexity analysis of the algorithm. We briefly outline these results in the following sections.

2.1 A fixed point evelope algorithm

For univariate contractive functions, Sikorski and Woźniakowski [13] developed a fixed point bisection envelope (BEN) method. This method constructs two envelope functions that interpolate already computed function values. Then, the set of all possible fixed points of functions that coincide at all evaluation points is given by the interval of uncertainty $[a, b]$, where a and b are the fixed points of the envelopes. Given the initial interval of uncertainty $[-1, 1]$, the method iteratively computes functions at the midpoints of intervals of uncertainty until the length of some interval is at most 2ε. Then the midpoint of the last interval is an ε-approximation to the fixed point.

Algorithm BEN:

Step 0 Given $\varepsilon > 0$. Let $a_0 = -1$, $b_0 = 1$, and $i := 0$.

Step 1 If $b_i - a_i \leq 2\varepsilon$, $(b_i + a_i)/2$ is an ε-approximation to x^*. Stop. Otherwise, go to Step 2.

Step 2 Let

$$x_{i+1} = (b_i + a_i)/2, \quad f_{i+1} = f(x_{i+1}).$$

If $f_{i+1} = x_{i+1}$, then x_{i+1} is the fixed point. Stop. If $f_{i+1} > x_{i+1}$, then let

$$\begin{aligned} a_{i+1} &= (f_{i+1} + qx_{i+1})/(1+q), \\ b_{i+1} &= \min(b_i, (f_{i+1} - qx_{i+1})/(1-q)). \end{aligned}$$

Otherwise, let

$$\begin{aligned} a_{i+1} &= \max(a_i, (f_{i+1} - qx_{i+1})/(1-q)), \\ b_{i+1} &= (f_{i+1} + qx_{i+1})/(1+q)). \end{aligned}$$

Let $i = i + 1$, and go to Step 1.

It was shown in [11] that for any $0 < q < +\infty$, the BEN method requires the minimal number of function evaluations to compute an ε-approximation to the fixed point of any f. This minimal number $\mathrm{m}(\varepsilon, q) < \lceil \log_2(2/\varepsilon) \rceil$.

We note that $\lceil \log_2(2/\varepsilon) \rceil$ is the number of function evaluations required by bisection. Obviously, $\mathrm{m}(\varepsilon, q)$ is much less than $n(\varepsilon, q)$, when q is close to 1.

2.2 Dimensional reduction scheme

A dimensional reduction scheme needed in our algorithm was presented in [12]. In the IE Algorithm, a sequence of volume-decreasing interior ellipsoids is constructed. If the radius of some ellipsoid is less than ε/α_n (see (2.6)), then the center of the ellipsoid is an ε-approximation of the fixed point, as guaranteed by Theorem 2.4. Otherwise, the ellipsoids become elongated. Once some ellipsoid is so flat that it can be well approximated by an $(n-1)$-dimensional hyperplane, the algorithm switches to this hyperplane to continue the fixed point approximation in the $n-1$ dimensional space. The IE algorithm repeats these bounding of fixed points and dimensional reduction steps in all dimensions except in the one dimensional case in which the BEN algorithm is used to approximate a one-dimensional fixed point.

Below we briefly outline a general flowchart of the algorithm with dimensional reduction scheme from [12].

We suppose that $(x-u)^T d = 0$ is the $(k-1)$-dimensional hyperplane, $k = n, \ldots, 3$, that approximates the k-dimensional inscribed ellipsoid, and that Q_k is an $n \times n$ orthogonal matrix in the form:

$$Q_k = \begin{bmatrix} I_{(n-k)\times(n-k)} & 0 \\ 0 & \bar{Q}_{k\times k} \end{bmatrix},$$

where $\bar{Q}_{k\times k}$ is a $k \times k$ orthogonal matrix which rotates the vector d onto the first coordinate axis of the k-dimensional space. Then the algorithm with dimensional reduction scheme can be described by the following general flowchart (Figure 1).

We make the following comments on the flowchart (Figure 1):

- Step 1 of the algorithm is realized by the IE algorithm described in Section 3.

$k := n,\ f^{[n]} = f$, and $Q := I_{n\times n}$;
while $k > 1$ **do**
begin
1. Find a $(k-1)$-dimensional hyperplane
$(x-u)^T d = 0$ such that
$|(x^{[k]*} - u)^T d| \leq \eta_{n-k+1}$,
where $x^{[k]*}$ is the fixed point of $f^{[k]}$, and
η_j's are termination parameters.
2. Find the matrix Q_k as defined above, and set:
$Q := Q_k Q$;
$k := k-1$;
$c_{n-k} := d^T u$;
$f^{[k]}(x) := P_k Q f(x) Q^T$;

end;
Use the BEN algorithm to find c_n such that $|x^{[1]*} - c_n| \leq \eta_n$.
return $Q^T [c_1, \ldots, c_n]^T$ as an ε-approximation of the fixed point x^*.

Figure 1: General flowchart of the dimensional reduction algorithm.

- It was shown in [12] that for

$$\eta_k = \frac{\varepsilon\sqrt{1-q^2}}{n}, \quad k = 1, \ldots, n-1, \tag{2.1}$$

$$\eta_n = \varepsilon/n \tag{2.2}$$

the algorithm computes vector $c = [c_1, \ldots, c_n]^T$ such that $\|Q^T c - x^*\| \leq \varepsilon$, i.e., $Q^T c$ is an ε-approximation to the fixed point x^* of $f(x)$.

2.3 A fixed point bounding lemma

We quote the following fundamental lemma for bounding fixed points [10, 11, 12].

Lemma 2.1 *We let $f \in F_n$, and suppose that $A \subseteq B^n(0,1)$ contains the fixed point x^*. Then for every $x \in A$, we have $x^* \in A \cap B^n(c, \gamma)$, where $c = x + \frac{1}{1-q^2}(f(x) - x)$ and $\gamma = \frac{q}{1-q^2}\|f(x) - x\|$.*

From Lemma 2.1, its proof [12], and the definition of $\bar{F}_n$ we have the following corollary.

Corollary 2.1 *We let f be any function in $\bar{F}_n$ and $A \subseteq [-1,1]^n$ be a polytope that contains x^*. Then, for any $x \in A$,*

$$x^* \in S = \{z \in A \ : \ a^T(z-b) \leq 0\}$$

with $a = \bar{x} - \bar{f}(\bar{x})$ and $b = (\bar{f}(\bar{x}) + q\bar{x})/(1+q)$, where

$$\bar{x} = \begin{cases} x & \text{if } x \in B^n(0,1), \\ \frac{x}{||x||} & \text{otherwise.} \end{cases}$$

Corollary 2.1 says that a smaller polytope, which is the intersection of A with the half space $a^T(z-b) \leq 0$, contains x^*.

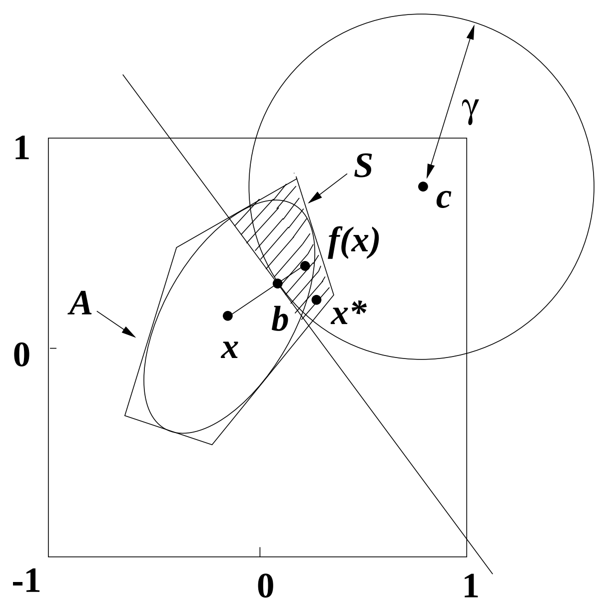

Figure 2: Polytope from Corollary 2.1.

2.4 Construction of γ-optimal inscribed ellipsoids

We let K be a convex body in R^k, $2 \leq k \leq n$. There is a unique inscribed ellipsoid E^* in K with the maximal volume [14]. An inscribed ellipsoid E is called γ-optimal, where $\gamma \in (0,1]$, if

$$\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(E^*)} \geq \gamma,$$

We let

$$\mu(K) = \max\{\mathrm{Vol}(E) \ : \ E \text{ is an ellipsoid and } E \subseteq K\},$$

and let $\bar{x}$ be the center of the maximal ellipsoid. For any hyperplane $H_a = \{x \ : \ a^T(x - \bar{x}) = 0\}$ passing through $\bar{x}$, we denote

$$K_{\pm} = \{x \ : \ x \in K, \ \pm a^T(x - \bar{x}) \leq 0\}$$

to be two bodies into which this hyperplane subdivides K. The following theorems from [14] give quantitative estimates of the volume reduction of the maximal inscribed ellipsoids and the γ-optimal inscribed ellipsoids.

Theorem 2.1

$$\mu(K_{\pm}) \leq 0.843\mu(K).$$

If we replace the maximal inscribed ellipsoid E^* with a γ-optimal inscribed ellipsoid E^*_γ in Theorem 2.1, we have

Theorem 2.2

$$\mu(K_{\pm}) \leq 0.843\gamma^{-2}\mu(K).$$

We now suppose that a polytope P given by

$$P = \{x \in R^k \ : \ a_j^T x \ \leq \ b_j, \ j = 1, \ldots, m\} \tag{2.3}$$

contains the fixed point x^*. We want to find a γ-optimal ellipsoid inscribed in P.

A k-dimensional ellipsoid centered at z can be represented as an affine transformation of the k-dimensional unit ball

$$E(X, z) = \{x = Xu + z \ : \ \|u\| \leq 1\},$$

where X is a positive definite matrix. Since

$$\phi(X, z) = \ln \mathrm{Vol}(E(X, z)) = \ln \det(X)$$

and $\ln\det(E(X,z))$ is a concave function on any convex domain [2], then the problem of finding a maximal volume ellipsoid inscribed in the polytope P can be formulated as the following convex programming problem [8]:

$$\begin{aligned} \min \quad & -\ln\, \det(X) \\ \text{subject to} \quad & X = X^T \geq 0, \\ & E(X,z) \subseteq P \end{aligned} \tag{2.4}$$

Several algorithms were proposed for solving (2.4) [3, 6, 8]. Probably the most efficient algorithm was given by Khachiyan and Todd [6]. They showed that a γ-optimal ellipsoid for the polytope P can be computed in at most

$$\mathrm{O}\left(m^{3.5}\ln\left[\frac{mW}{\ln(1/\gamma)}\right]\ln\left[\frac{k\ln W}{\ln(1/\gamma)}\right]\right) \tag{2.5}$$

arithmetic operations, where W is an a priori known ratio of the radii of two Euclidean balls, the first of which is circumscribed about P and the second inscribed in P. They noted that for the method of inscribed ellipsoids one can assume without essential loss of generality that $\gamma = 0.99$, $W = 3k$, $m = \mathrm{O}(k\ln k)$.

2.5 John's theorem

The following results are utilized in the complexity analysis of the IE algorithm.

Theorem 2.3 *(The Löwner-John ellipsoid[4]). For each convex body K in R^k, there exist a point x and a linear transformation L such that*

$$x + L(B^k(0,1)) \subset K \subset x + kL(B^k(0,1)).$$

The ellipsoid $E = x + L(B^k(0,1))$ is the ellipsoid of maximum volume inscribed in K, and $E^d = x + kL(B^k(0,1))$ is the homotetic dilatation of the ellipsoid E by the factor k.

According to the above theorem, the dilatated ellipsoid E^d contains the set K.

A similar theorem holds for the γ-optimal ellipsoids. Namely, we have

Theorem 2.4 *(see [5, 14]) If $E_\gamma \subset K$ is a γ-optimal ellipsoid inscribed in a convex body $K \subset R^k$, then*

$$K \subset \alpha_k E_\gamma,$$

where $\alpha_k E_\gamma$ is the homotetic dilatation of E_γ by the constant

$$\alpha_k = \frac{1 + 6\sqrt{1-\gamma}}{\gamma} k. \tag{2.6}$$

3 Inscribed Ellipsoid Algorithm

Now we are in a position to present the inscribed ellipsoid algorithm. In this algorithm, a set of hyperplanes is constucted to form a polytope that contains the fixed point x^*. In each step we find a γ-optimal ellipsoid inscribed in the polytope. If the radius of the ellipsoid is less than ε/α_k, then the center of the ellipsoid is an ε-approximation of x^*. If the smallest axis of the ellipsoid is so small that the ellipsoid can be well approximated by a hyperplane, the dimensional reduction scheme is carried out. Otherwise, we find a hyperplane passing through the center of the ellipsoid (Corollary 2.1) and decide which half space contains x^*. We modify the polytope by adding this extra constraint. If the number of hyperplanes exceeds some preset number $N(k)$ for the dimension k, we dilatate the ellipsoid by the factor α_k. Then we construct the smallest box, which encloses the dilated ellipsoid, by finding $2k$ hyperplanes pairwisely orthogonal to each axis of the dilated ellipsoid. By Theorem 2.4, this box encloses the original polytope and therefore contains the fixed point x^*. Then we restart the algorithm from this box.

The inscribed ellipsoid algorithm can be formulated as follows:

Algorithm IE:

Step 0 Given constants ε, $\gamma \in (0,1)$. Let $P^{(0)} = [-1,1]^n$ (Observe that $2n$ linear constraints uniquely represent $P^{(0)}$). Let $k := n$ (k is the current dimension), $i = 0$, $f^{[n]} = f$, and Q be the $n \times n$ identity matrix.

Step 1 If $k = 1$, go to Step 5. Otherwise, construct a γ-optimal inscribed ellipsoid E_i in $P^{(i)}$

Step 2 (Termination check) If the radius of E_i is less than ε/α_k, the center of E_i is an ε-approximation of x^*. Stop; If the length $\alpha_k\sqrt{\lambda_1(E_i)}$ of the smallest semi-axis of the dilataed ellipsoid E_i^d satisfies

$$\alpha_k \sqrt{\lambda_1(E_i)} \leq \eta_k = \varepsilon \sqrt{1-q^2}/n, \tag{3.1}$$

where $\lambda_1(E_i)$ is the smallest eigenvalue of the matrix defining E_i, then carry out the dimensional reduction scheme. Let $k = k - 1$ and go to Step 1. Otherwise, go to Step 3.

Step 3 Evaluate $f(x)$ at the center of E_i to decide which part of the polytope $P^{(i)}$ contains x^*. Find a half space

$$h_i = \{x \ : \ p_i^T x \ \leq \ a_i\}$$

such that $h_i \cap P^{(i)}$ contains x^* (See Corollary 2.1). Add h_i to the set of constaints for $P^{(i)}$ to form $P^{(i+1)}$.

Step 4 If $i \geq N(k)$, find $2k$ hyperplanes

$$l_j : \ \ p_j^T x \ = \ a_j, \quad j = 1, \ldots, 2k$$

that bound $P^{(i+1)}$. Let

$$P^{(0)} = \{x \in R^k \ : \ p_j^T x \ \leq \ a_j, \ j = 1, \ldots, 2k\}.$$

Let $i = 0$ and go to Step 1. Otherwise, let $i = i + 1$ and go to Step 1.

Step 5 Use Algorithm BEN with $\varepsilon = \varepsilon/n$ to find c_n. Then $Q^T[c_1, \ldots, c_n]$ is an ε-approximation to the fixed point x^*.

Below we clarify Steps 2 and 4 of the algorithm.

Termination condition (Step 2)

We enter the dimensional reduction stage whenever we have already approximated one component of the fixed point to within an error of at most $\varepsilon\sqrt{1-q^2}/n$, see (2.1). The polytope containing the fixed point is a subset of the dilatated ellipsoid E_i. Then we know that the distance of the fixed point to the center of E_i as measured along the smallest axis is at most $\alpha_k \eta_i$.

Dilatation of ellipsoids (Step 4)

To limit the cost in (2.5) we need to control the number of constraints defining $P^{(i)}$ in our algorithm. If the number of hyperplanes exceeds $N(k)$, we find $2k$ new hyperplanes forming a new polytope box which contains x^*. This is accomplished by the following.

We let E be the γ-optimal inscribed ellipsoid from Step 1,

$$E = \{Xu + z \ : \ \|u\| \leq 1\}$$

According to Theorem 2.4, the ellipsoid

$$E^d = \{(\alpha_k X)u + z \ : \ \|u\| \leq 1\}$$

contains $P^{(i)}$, and x^* as well. Thus, the new polytope is obtained by bounding E^d by $2k$ hyperplanes pairwisely orthogonal to the corresponding axis of E^d.

We note that E^d may not be contained in $[0,1]^k$. If we need to compute $f(x)$ for x outside $B^k(0,1)$ we do use the extension $\bar{f}(x)$ of $f(x)$ as defined in (1.2).

4 Complexity analysis

In this section, we give a quantitative estimate of the computational cost of the IE algorithm. We assume that $\gamma = 0.999$, and denote

$$R = 0.843\gamma^{-2} \approx 0.845. \tag{4.1}$$

4.1 Cost of the "pure" inscribed ellipsoid algorithm

In this section, we assume that $N(k)$ can be arbitrarily large for each dimension k, so that the ellipsoid dilatation step does not take place. In this case, we call the IE algorithm as the "pure" inscribed ellipsoid algorithm.

We let $B(z,r)$ be the k-dimensional ball with radius r centered at z. Then the volume of $B(z,r)$ is

$$\mathrm{Vol}(B(z,r)) = r^k \omega_k,$$

where

$$\omega_k = \frac{\pi^{k/2}}{\Gamma(1+\frac{k}{2})},$$

and $\Gamma(\cdot)$ is the Gamma function. Obviously, from the IE algorithm, we have

$$E_0 = B^k(0,1) \subseteq [-1,1]^k,$$

Thus, according to Theorem 2.2 and (4.1), after i steps, we have

$$\mathrm{Vol}(E_i) \le R^i \mathrm{Vol}(E_0) = R^i \omega_k. \tag{4.2}$$

On the other hand, since the largest ball inscribed in E_i has the radius $\sqrt{\lambda_1(E_i)}$, we have

$$\mathrm{Vol}(E_i) \ge \left(\sqrt{\lambda_1(E_i)}\right)^k \omega_k \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\left(\sqrt{\lambda_1(E_i)}\right)^k \le R^i,$$

which implies that

$$\sqrt{\lambda_1(E_i)} \le R^{i/k}. \tag{4.4}$$

From (2.1) and Theorem 2.4, we require that

$$\alpha_k \sqrt{\lambda_1(E_i)} \;\le\; \eta_k \;=\; (\varepsilon/n)\sqrt{1-q^2}. \tag{4.5}$$

before the dimensional reduction scheme is carried out. This is satisfied whenever

$$\alpha_k R^{i/k} \;\le\; (\varepsilon/n)\sqrt{1-q^2}.$$

Hence, we have

$$\begin{aligned} i \;&\ge\; \frac{k \ln \frac{\alpha_k n}{\varepsilon\sqrt{1-q^2}}}{-\ln R} \\ &=\; (-\ln R)^{-1} k \left(\ln n + \ln k + \ln \frac{1}{\varepsilon} + (1/2) \ln \frac{1}{1-q} \right. \\ &\quad \left. + \ln \frac{1+6\sqrt{1-\gamma}}{\gamma} + (1/2) \ln \frac{1}{1+q} \right). \end{aligned} \tag{4.6}$$

We denote H and δ as

$$H = (-\ln R)^{-1} \approx 5.925, \qquad \delta = \ln \frac{1+6\sqrt{1-\gamma}}{\gamma} \approx 0.175. \tag{4.7}$$

Then, we can take

$$i = \left\lceil Hk \left(\ln n + \ln k + \ln \frac{1}{\varepsilon} + (1/2) \ln \frac{1}{1-q} + \delta + \ln \frac{1}{2\sqrt{1+q}} \right) \right\rceil. \tag{4.8}$$

From (4.6) and (4.8), the total number of steps in dimensions $k = 2, \ldots, n$ is

$$
\begin{aligned}
S &\leq \sum_{k=2}^{n} \left[Hk \left(\ln n + \ln k + \ln \frac{1}{\varepsilon} + \frac{1}{2} \ln \frac{1}{1-q} + \delta + (1/2) \ln \frac{1}{1+q} \right) + 1 \right] \\
&\leq H \left(\frac{1}{2} n(n+1) \ln n + \int_2^n (x \ln x) dx + \frac{1}{2} n(n+1) \ln \frac{1}{\varepsilon} \right. \\
&\quad \left. + \frac{1}{4} n(n+1) \ln \frac{1}{1-q} + \frac{1}{2} n(n+1)\delta + \frac{1}{4} n(n+1) \ln \frac{1}{1+q} \right) + n \\
&\leq \frac{1}{2} Hn(n+1) \left(\ln n + \ln \frac{1}{\varepsilon} + \frac{1}{2} \ln \frac{1}{1-q} + \delta + \frac{1}{2} \ln \frac{1}{1+q} \right) \\
&\quad + H \left(\frac{1}{2} n^2 \ln n - \frac{1}{4} n^2 - 2 \ln 2 + 1 \right) + n
\end{aligned} \tag{4.9}
$$

We let $C_k(f)$ be the cost of one function evaluation and $C_k(E_\gamma)$ be the cost of finding a γ-optimal inscribed ellipsoid, in k dimensional space, respectively. Then, the total cost of the "pure" inscribed ellipsoid algorithm is

$$
\text{Cost} \leq \mathrm{m}(\varepsilon/n, q) \mathrm{C}_1(f) + S \left(\mathrm{C}_n(f) + \mathrm{C}_n(E_\gamma) \right), \tag{4.10}
$$

where $\mathrm{m}(\varepsilon, q)$ is defined in (1.8), with the assumption

$$
\mathrm{C}_k(f) \leq \mathrm{C}_n(f), \qquad \mathrm{C}_k(E_\gamma) \leq \mathrm{C}_n(E_\gamma).
$$

for $k = 2, \ldots, n-1$.

4.2 Cost of the IE algorithm with "cycles".

From (4.10), we know that if the cost of function evaluations is moderate, then the cost of finding γ-optimal inscribed ellipsoids may be a significant part of the total cost. Formula (2.5) implies that the cost of finding γ-optimal inscribed ellipsoids depends on m, the number of constraints of the convex programming (2.4). Since m is increased by one every time a new constraint is added in Step 3 of the IE algorithm, then the $C_k(E_\gamma)$ can be very large if $N(k)$ is not bounded.

In this section, we assume that

$$
N(k) = Ck \ln k, \tag{4.11}
$$

where C is a constant independent of k. In this case, the number of the constraints m in (2.4) is

$$
m \leq 2k + N(k) = \mathrm{O}(k \ln k). \tag{4.12}
$$

We construct the γ-optimal inscribed ellipsoids in cycles of $N(k)$ steps. After each $N(k)$ steps we dilatate the resulting ellipsoid by the factor α_k, and restart the construction from the box containing the dilatated ellipsoid. Then, from Theorem 2.2, we conclude that after s cycles of the constructions, we get

$$\begin{aligned}
\mathrm{Vol}(E^d_{sN(k)})) &= \alpha_k^k R^{N(k)} \mathrm{Vol}(E^d_{(s-1)N(k)}) \\
&\quad \cdots\cdots \\
&\le \alpha_k^{sk} R^{sN(k)} \mathrm{Vol}(E_0) \\
&= \alpha_k^{sk} R^{sN(k)} \omega_k
\end{aligned} \tag{4.13}$$

Then as in (4.3) we have

$$\mathrm{Vol}(E^d_{sN(k)}) \ge \left(\sqrt{\lambda_1(E^d_{sN(k)})}\right)^k \omega_k. \tag{4.14}$$

By combining (4.11), (4.13) and (4.14), we get

$$\sqrt{\lambda_1(E^d_{sN(k)})} \le \alpha_k^s R^{sN(k)/k} = \alpha_k^s R^{Cs\ln k}. \tag{4.15}$$

The deflation of dimension is carried out when the smallest axis of the ellipsoid $E^d_{sN(k)}$ is at most $\varepsilon\sqrt{1-q^2}/n$. This is satisfied whenever

$$\alpha_k^s R^{Cs\ln k} \le \frac{\varepsilon\sqrt{1-q^2}}{n}. \tag{4.16}$$

From (4.7) and (4.16), we have

$$s \ge \frac{\ln n + \ln\frac{1}{\varepsilon} + (1/2)\ln\frac{1}{1-q} + (1/2)\ln\frac{1}{1+q}}{(C\ln\frac{1}{R} - 1)\ln k - \delta} \tag{4.17}$$

Hence,

$$s = \frac{\ln\frac{1}{\varepsilon} + (1/2)\ln\frac{1}{1-q} + \ln n + (1/2)\ln\frac{1}{1+q}}{(C\ln\frac{1}{R} - 1)\ln k - \delta} + p \tag{4.18}$$

for some $p \in [0,1)$. In this case, the total number of steps S_k before the dimensional reduction is carried out is

$$\begin{aligned}
S_k &= sCk\ln k \\
&= \frac{C}{C\ln\frac{1}{R} - 1 - \delta/\ln k} \cdot k\left(\ln\frac{1}{\varepsilon} + (1/2)\ln\frac{1}{1-q} + \ln n + (1/2)\ln\frac{1}{1+q}\right) \\
&\quad + pCk\ln k
\end{aligned} \tag{4.19}$$

When ε is small and q is close to 1, the dominant term in (4.19) is

$$k\left(\ln\frac{1}{\varepsilon}+(1/2)\ln\frac{1}{1-q}+\ln n\right).$$

We denote $K(C,k)$ as

$$K(C,k)=\frac{C}{C\ln\frac{1}{R}-1-\delta/\ln k}. \tag{4.20}$$

We need to choose C such that $K(C,k)$ is reasonably small. We note that

$$K(C,k)\le K(C,2),\quad \text{for } k\ge 2.$$

Figure 3 shows the relationship of $K(C,2)$ and C. The graph of $K(C,2)$ indicates that $18.2\ge K(C,2)\ge 11.7$ for $C\in[11,15]$.

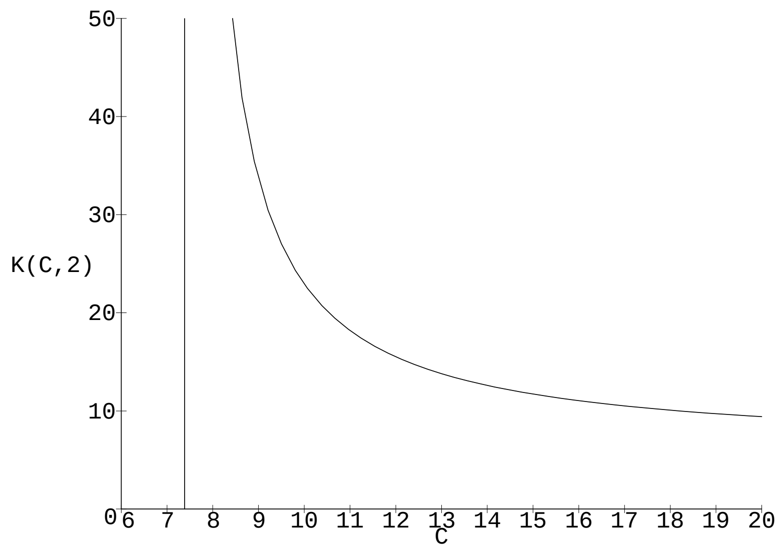

Figure 3: Illustration of the dependency of $K(C,2)$ on C.

Total cost of the IE algorithm with "cycles"

Based on the above analysis, we can give the total cost of the IE algorithm.

From (2.5) and (4.12), the cost of constructing ellipsoids is

$$\begin{aligned}\mathrm{C}_k(E_\gamma) &\le \mathrm{O}\left(k^{3.5}(\ln k)^{4.5}(\ln k+\ln\ln k)\right)\\ &\approx \mathrm{O}\left(k^{3.5}(\ln k)^{5.5}\right),\end{aligned} \tag{4.21}$$

Hence, the total cost of the IE algorithm is

$$\text{Total Cost} = \mathrm{m}(\varepsilon/n, q)\mathrm{C}_1(f) + \sum_{k=2}^{n} S_k\left(\mathrm{C}_k(f) + \mathrm{C}_k(E_\gamma)\right) \tag{4.22}$$

When ε is small and q is close to 1, we have

$$\begin{aligned}\text{Total Cost} \quad &\approx \quad \frac{1}{\ln 2}\left(\ln\frac{2n}{\varepsilon}\right)\mathrm{C}_1(f) \\ &\quad + \frac{1}{2}K(C,2)n^2\left(\ln\frac{1}{\varepsilon} + \frac{1}{2}\ln\frac{1}{1-q} + \ln n\right)\left(\mathrm{C}_n(f) + \mathrm{C}_n(E_\gamma)\right) \quad (4.23)\end{aligned}$$

4.3 Conclusions

From the above analysis, we conclude that:

1. If the cost of each function evaluation is much larger than the ellipsoid construction cost, then the total cost is $\mathrm{O}(n^2(\ln\frac{1}{\varepsilon} + \ln\frac{1}{1-q} + \ln n))$, i.e., the IE algorithm is asymptotically of the same cost as the centroid method [11].

2. If $\mathrm{C}_k(f)$ is smaller or about the same as $\mathrm{C}_k(E_\gamma)$, then the total cost depends on n, ε, q as:
$$\mathrm{O}\left((n\ln n)^{5.5}(\ln\frac{1}{\varepsilon} + (1/2)\ln\frac{1}{1-q} + \ln n)\right).$$

3. It is an interesting problem to find the arithmetic complexity of finding γ-optimal ellipsoids. The estimate (2.5) is the best result known to us at this point.

4. The algorithm described in [6] handles linear constraints. Can this method be generalized to quadratic constraints (as needed in our algorithm)? This may result in faster volume reduction of interior ellipsoids.

5. We conjecture that the bound
$$\mathrm{O}\left(n^2(\ln\frac{1}{\varepsilon} + \ln\frac{1}{1-q} + \ln n)\right)$$
on the number of function evaluations is optimal to within a multiplicative constant. Affirmative proof of this would imply almost optimality of the centroid algorithm. We remark that the constant $H = 5.925$ in the "pure" IE algorithm is about 2.7 times larger than the constant $H = -\frac{1}{\ln(1-e^{-1})} \approx 2.18$ in the centroid algorithm.

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