FoDA L9

Linear Algebra Review #2

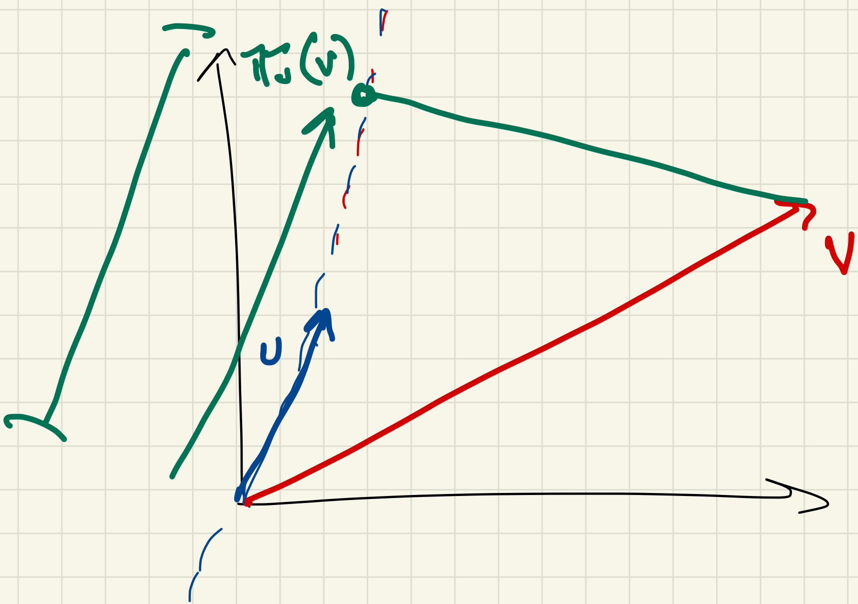
Norms, Linear Independence, Rank

Vectors $v \in \mathbb{R}^d$

$v = (v_1, v_2, \dots v_d)$

matrices $A \in \mathbb{R}^{n \times d}$

$$\begin{bmatrix} A_{11} & A_{12} & & \\ A_{21} & & & \\ & & & A_{n,d} \end{bmatrix}$$



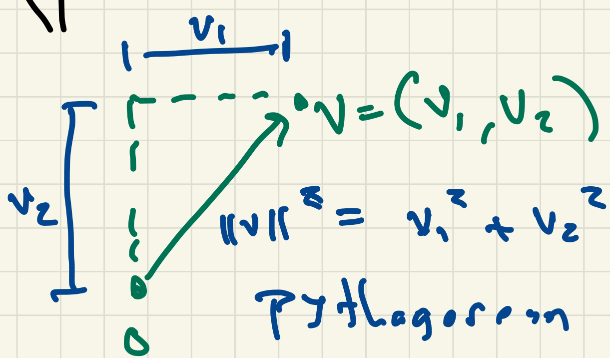
Norm $\|u\| = 1$

length$(u) = 1$

$\langle u, v \rangle = $ length$(\pi_u(v))$

Norm of vector $v \in \mathbb{R}^d$ (Euclidean)

$$\|v\|_2 = \sqrt{\sum_{i=1}^{d} v_i^2} = \sqrt{v_1 v_1 + v_2 v_2 + \cdots v_d v_d} = \sqrt{\langle v, v \rangle}$$



$$\|v\|^2 = v_1^2 + v_2^2$$

Pythagorean

if $\|v\| = 1$

v <u>unit</u> vector

normalize $v \to \hat{v}$ $\quad \|\hat{v}\| = 1$

$$\hat{v} = \left(v_1 / \|v\|,\ v_2 / \|v\|,\ \ldots,\ v_d / \|v\| \right)$$

$$\sqrt{\sum_{i=1}^{d} \frac{v_i^2}{\|v\|^2}} = \sqrt{\frac{1}{\|v\|^2} \left(\sum_{i=1}^{d} v_i^2 \right)} = \frac{\|v\|}{\|v\|} = 1$$

p-Norm of a vector $\quad p \in [1, \infty)$

$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p \right)^{1/p}$$

$v = [-1, 2, 7, 5]$

$$\|v\|_1 = \sum_{i=1}^{d} |v_i|$$

$\|v\|_1 = |-1| + 2 + 7 + 5 = 15$

$\neq 13$

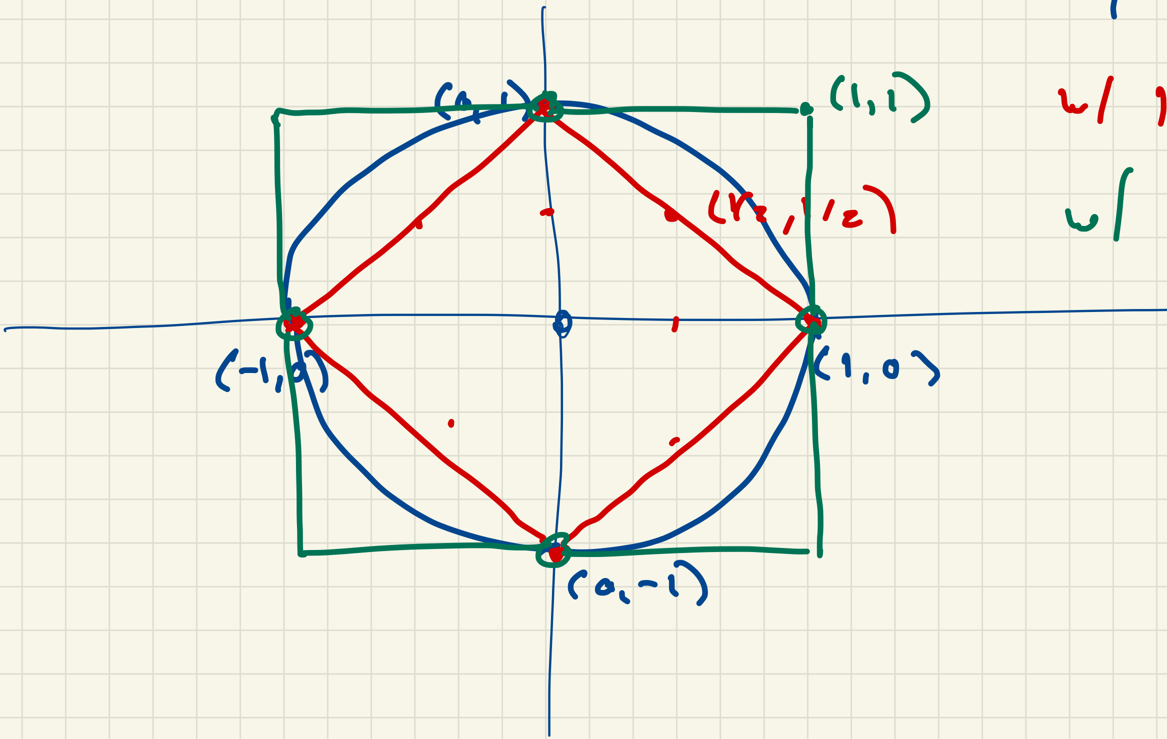
$$\|v\|_\infty = \max_{i=1..d} |v_i|$$

$\|v\|_\infty = 7$

$\|v\|_2 = \sqrt{1^2 + 2^2 + 7^2 + 5^2}$

$\sqrt{1 + 4 + 49 + 25}$

$= \sqrt{79} \qquad = 8.88$



set of all vectors

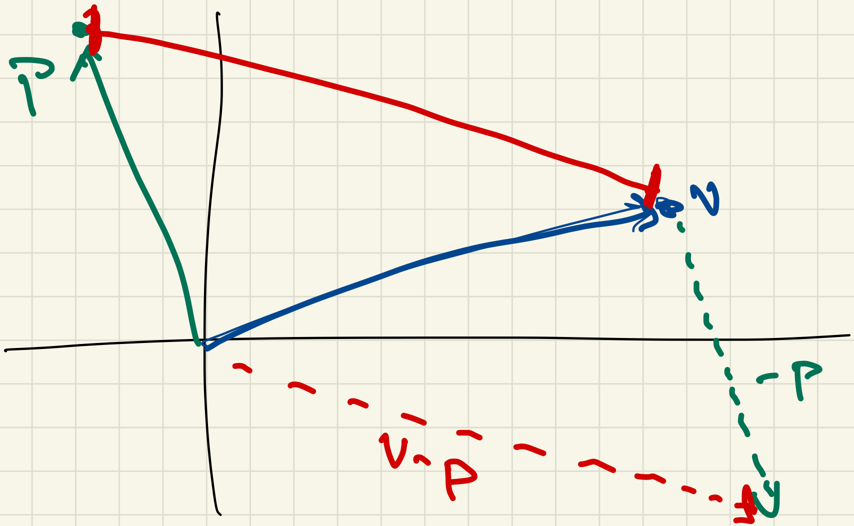
w/ $\|v\|_2 = 1$

w/ $\|v\|_1 = 1$

w/ $\|v\|_\infty = 1$

Distance

$$L_p(v, p) = \|v - p\|_p$$



Norm of a matrix $A \in \mathbb{R}^{n \times d}$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \left(\sum_{j=1}^{d} A_{ij}^2 \right)} = \sqrt{\sum_{i=1}^{n} \left(\|a_i\|^2 \right)} \qquad A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Frobenius

$$\|A\|_2 = \max_{\substack{x \in \mathbb{R}^d \\ \|x\| \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\substack{y \in \mathbb{R}^n \\ \|y\| \neq 0}} \frac{\|yA\|}{\|y\|}$$

spectral

$$= \max_{\substack{u \in \mathbb{R}^d \\ \|u\| = 1}} \|Au\|_2$$

$$\|A\|_F \geq \|A\|_2$$

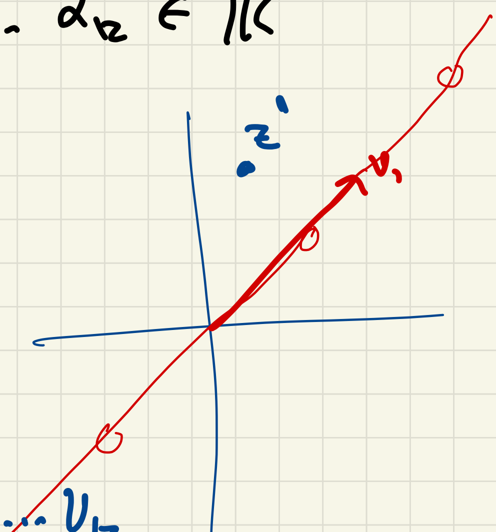
Linear Indepence

$k < d$

fix k vectors $v_1, v_2, \dots v_k \in \mathbb{R}^d$

any k scalars $\alpha_1, \alpha_2 \dots \alpha_k \in \mathbb{R}$

$$z = \sum_{j=1}^{k} \alpha_j v_j \in \mathbb{R}^d$$



any z can be written

w/ some $\alpha = (\alpha_1, \dots \alpha_k)$

linear dependend on $v_1 \dots v_k$

if z' $\boxed{\text{cannot}}$ be written this way

linearly independent on $v_1 \dots v_k$

Set

$$\text{span}(V) = \left\{ z \;\middle|\; z = \sum_{i=1}^{k} \alpha_i v_i \,,\ \alpha_i \in \mathbb{R} \right\}$$

if $\text{span}(V) = \mathbb{R}^d$

then V basis for $\mathbb{R}^d$

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \qquad \in \mathbb{R}^3$$

$$z_1 = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} \qquad z_2 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

linearly independent

$$z_1 = \alpha_1 v_1 + \alpha_2 v_2$$

linearly dependent

$$1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{matrix} 1 \\ 3 \\ 4 \end{matrix} + \begin{matrix} -4 \\ -8 \\ -2 \end{matrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} = z_1$$

$$z_2 = \alpha_1' v_1 + \alpha_2' v_2$$

$$\begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

Set of vectors $X = \{x_1, \dots x_n\}$

linearly independent

each x_i no way to choose

scalars $\{\alpha_1 \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_n\}$

s.t.

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j x_j$$

$$A = \begin{bmatrix} 0 & 1.5 \\ 2 & 0.5 \\ 1 & 1 \end{bmatrix}$$