Markov Chains

Note Title 4/6/2016

A5: Due April 15

(L1) Only current pos, don't worry about the past

(L2) Don't worry about distant future

↳ go one step at a time

(L3) In the limit, Everyone has perfect karma.

Markov Chain

V: node set $|V| = m$

P: $m \times m$ matrix, prob. transition matrix

q: q_0 initial state

$$q_{i+1} = P q_i$$

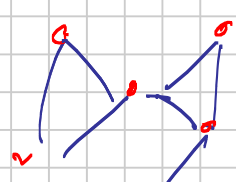
(L1) "Markov"

q_i = probability distribution on V.

Two Views of MCs.

1. Random Walk (in exactly one state)

$q = [0,0,0,\ldots,1,0,0,0]^T$



Metropolis Algo. MCMC $q_i = f(q_{i-1})$

7. Probability Distribution of Random walks.

q_i: dense $\qquad q_i = P^i q_0 = P q_{i-1}$

Interested in q_n for very large n.

$$n \to \infty \qquad q_n = q_*$$

<u>Ergodic</u> MC is ergodic if

$\exists\, t$ s.t. for all $n \geq t$, then

P^n is positive in every entry.
(non-zero)

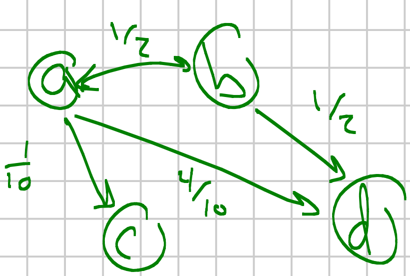
$$q_n = P^n q_0$$

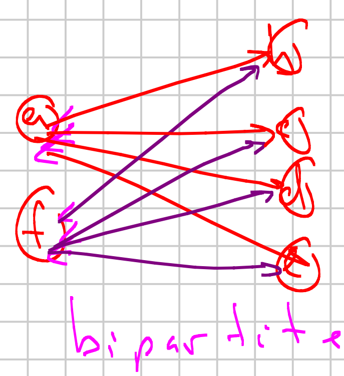
Not ergodic only if

- cyclic
- absorbing and transient
- not connected.

depend on q_0

$$\begin{pmatrix} 0 & 1/2 & 1/2 & \cdots & 1/2 & 0 \\ 1/4 & & & & & 1/4 \\ 1/4 & & & & & 1/4 \\ 1/4 & & & & & 1/4 \\ 1/4 & & & & & 1/4 \\ 0 & 1/2 & 1/2 & \cdots & 1/2 & 0 \end{pmatrix}$$





If ergodic: only need P

- Exists $P^* = P^n$ as $n \to \infty$

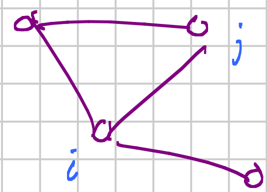
  exists $q_* = P^* q$ (any q)

  $\longrightarrow$ For all q_0 if we run MC long enough $q_i \to q_*$ (L2)

  $q_i = P q_{i-1}$ or q_i RW q_{i-1}

- $q_* = \overbrace{P P^*}^{P^{*+1}} q = P \underbrace{P^* q}_{q_*} \Rightarrow \boxed{q_* = P q_*}$

  $P_{ji}\, q_*(j) = P_{ij}\, q_*(i)$

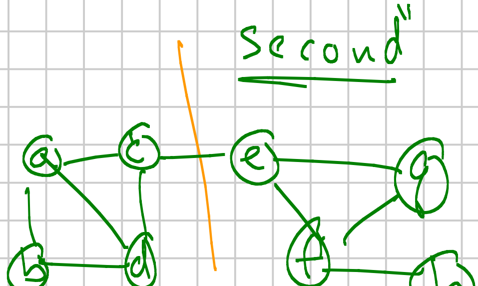


- q_* first eigenvector of P.

  [V, L] = eig(P)

  v1 = V(:, 1)   qstar = v1 / sum(v1)

"Second" eigenvalue $\Rightarrow$ rate of convergence



$v_1 = q_* = (0.15, 0.1, 0.15, 0.15, 0.15, 0.1, 0.05)^T$

$\lambda_2 = 0.875 \Rightarrow$ smaller = faster

Metropolis Algo.

MRRTT
1953

Hastings 1970

state space V

$v \in V$: weight $w(v)$, $\sum_{v \in V} w(v) = W$

$$q_*(v) = \frac{w(v)}{W}$$

Metropolis Alg.

$q = \xi_0 = [0,0,0,1,0,0,0]^T \equiv v \in \mathbb{R}^d$

repeat

$u \sim K(v_i, \cdot)$ $\quad u$ = neighb of v

if $(w(u) \geq w(v_i))$

$v_{i+1} = u$

else

w.p. $\frac{w(u)}{w(v_i)}$ set $v_{i+1} = u$

else

$v_{i+1} = v_i$

$$V = \{\overbrace{v_1, v_2}^{\text{burn in}}, \ldots, v_n\}$$