L16: SVD and Relatives

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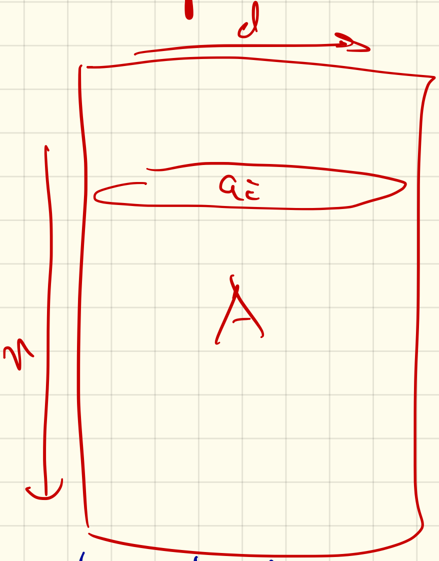
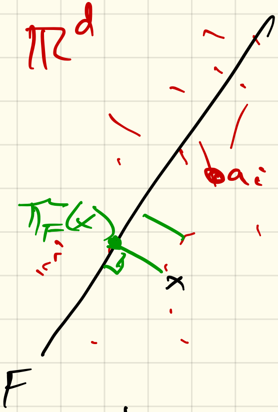
Input $A \in \mathbb{R}^{n \times d}$



$k \ll d$



$a_i \in \mathbb{R}^d$

Goal Find subspace F

F k-dimensional

$$\pi_F(x) = \underset{p \in F}{\operatorname{argmin}} \|p - x\|$$

Must be data matrix

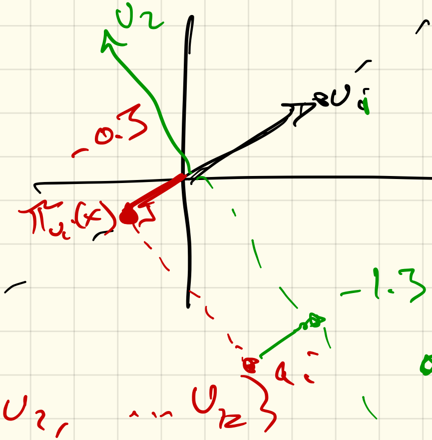
$$F^* = \underset{\substack{F \\ \operatorname{rank}(F) = k \\ 0 \in F}}{\operatorname{argmin}} \sum_{a_i \in A} \|x - \pi_F(x)\|^2$$

every coord same units

If $F = u_i$ where u_i unit vector

$k=1$ $\quad \|u_i\|=1$

$$\pi_{u_i}(x) = \langle u_i, x \rangle u_i$$



$\alpha_i \in U_F = (-0.3, -1.3)$

F via basis $U_F = \{u_1, u_2, \dots u_k\}$

$\|u_i\| = 1$

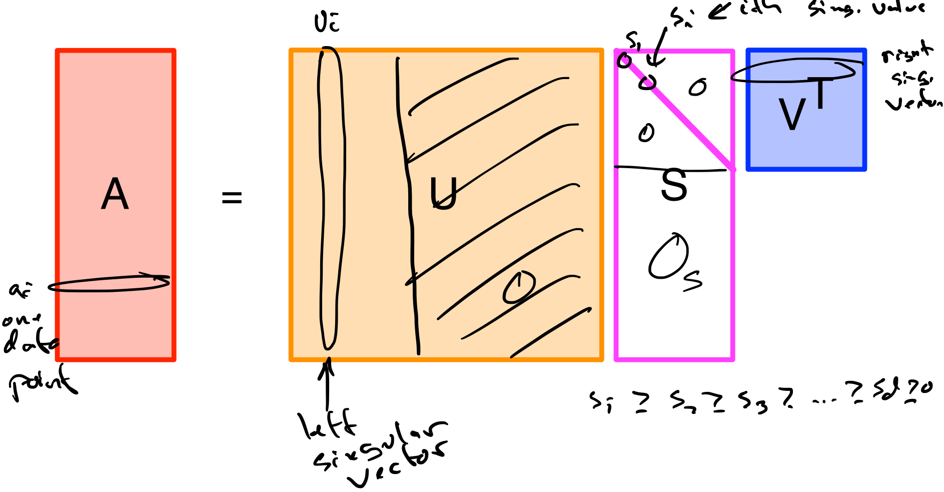
$\langle u_i, u_j \rangle = 0 \quad i \neq j$

For any $p \in F \Rightarrow p = \sum_{i=1}^{k} \alpha_i u_i$ where $\alpha_i = \langle u_i, p \rangle$

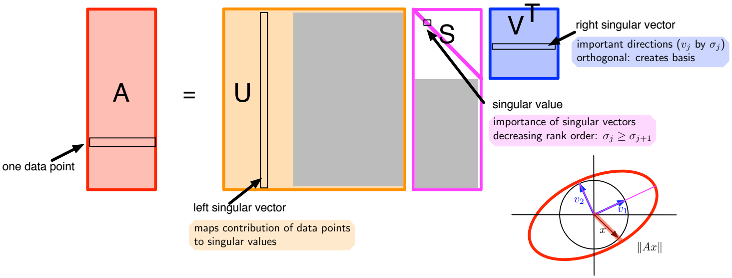
$p \in U_F$ as $(\alpha_1, \alpha_2 \dots \alpha_k) \in \mathbb{R}^k$

Singular Value Decomposition

For $n \times d$ matrix A, define $[U, S, V] = \text{svd}(A)$ so that $USV^T = A$.



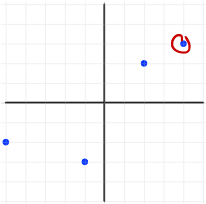
Singular Value Decomposition



Tracing a Point through SVD

Consider a matrix

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix},$$



and its SVD $[U, S, V] = \mathrm{svd}(A)$:

$$U = \begin{pmatrix} -0.6122 & 0.0523 & 0.0642 & 0.7864 \\ -0.3415 & 0.2026 & 0.8489 & -0.3487 \\ 0.3130 & -0.8070 & 0.4264 & 0.2625 \\ 0.6408 & 0.5522 & 0.3057 & 0.4371 \end{pmatrix} S = \begin{pmatrix} 8.1655 & 0 \\ 0 & 2.3074 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V = \begin{pmatrix} -0.8142 & -.5805 \\ -0.5805 & 0.8142 \end{pmatrix}$$

Tracing a Point through SVD

$Ax = USV^T x$

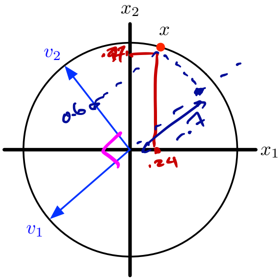
$V^T x$

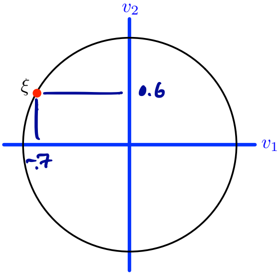
$SV^T x$

$x = (0.243, 0.97)$, then what is $\xi = V^T x$?

$Ax = USV^T x$

$$V = \begin{array}{c} v_1 \\ v_2 \end{array} \begin{pmatrix} -0.8142 & -.5805 \\ -0.5805 & 0.8142 \end{pmatrix}$$

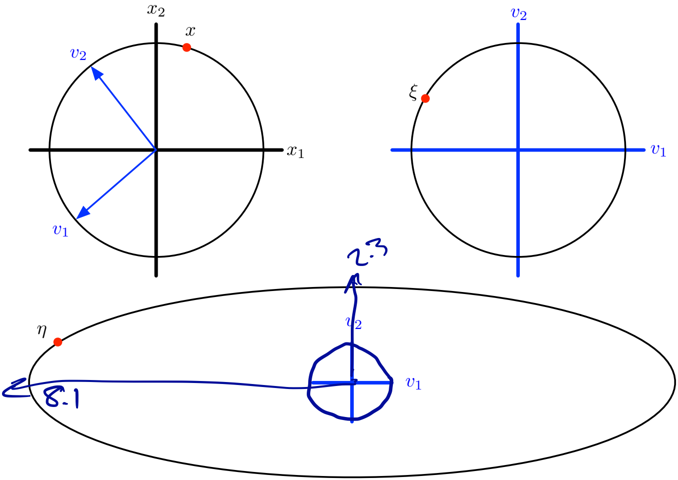




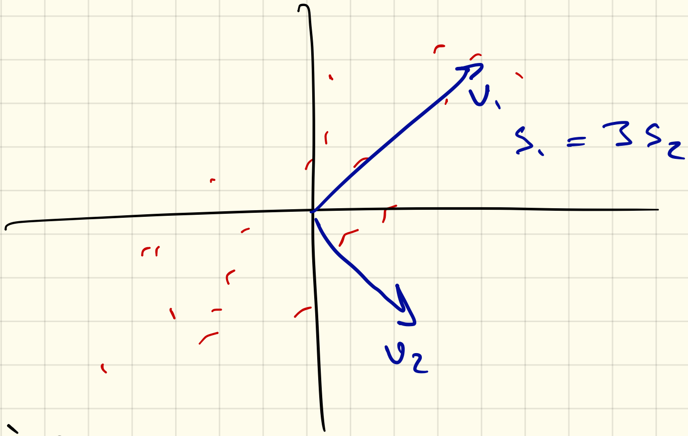
Tracing a Point through SVD

$x = (0.243, 0.97)$, then what is $SV^Tx = S\xi$?

$$V = \begin{pmatrix} -0.8142 & -.5805 \\ -0.5805 & 0.8142 \end{pmatrix}$$



Best Rank-k Approx A



$v_1 \leftarrow$ top right sing. vector of A

$$= \underset{\substack{\|v\|=1 \\ v \in \mathbb{R}^d}}{\operatorname{argmax}} \left(\|Av\|^2 = \sum_{a_i \in A} \|a_i^T v\|^2 = \sum_{a_i \in A} \langle a_i, v \rangle^2 \right)$$

$$s_j^2 = \|A v_j\|^2 = \sum_{a_i \in A} \langle a_i, v_j \rangle^2$$

$$\|A\|_F^2 = \sum_{a_i \in A} \|a_i\|^2 = \sum_{j=1}^{d} \|A v_j\|^2 = \sum_{i=1}^{d} s_i^2$$

$$F^* = \arg\min_{\substack{F \\ \mathrm{rank}(F)=k}} \sum_{a_i \in A} \| a_i - \pi_F(a_i) \|^2$$

$$= \arg\max_{\substack{F \\ \mathrm{rank}(F)=k}} \sum_{a_i \in A} \| \pi_F(a_i) \|^2$$

$$F_\perp^* = \mathbb{R}^d \setminus F^*$$

$(d-k)$-dim

$$= F_\perp^* = \arg\min_{\substack{F \\ \mathrm{rank}(F)=d-k}} \sum_{a_i \in A} \| \pi_F(a_i) \|^2$$

$\rightarrow F$ by

span $v_1, v_2 \ldots v_k$

set

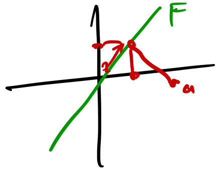
$v_j = V_j$

jth right

singr

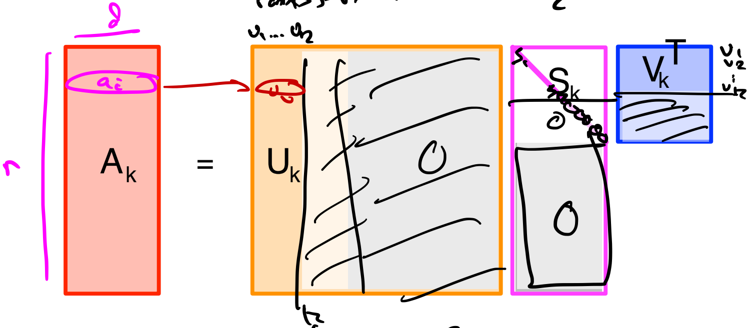
vector

Best Rank k-Approximation



$$A_k^* = \arg\min \| A - A_k \|_F^2$$

$$\arg\min_{\mathrm{rank}(B)=k} \| A - B \|_2^2$$



$$A_k = \underbrace{\sum_{j=1}^{k} u_j s_j v_j^T}_{n \times d}$$

A

$\alpha_1 = (a, v_1)$

$\alpha_2 = (a, v_2)$

$a_i \in U_{F_k} \Rightarrow u_i$

So far F most contain origin

Principal Component Analysis (PCA)

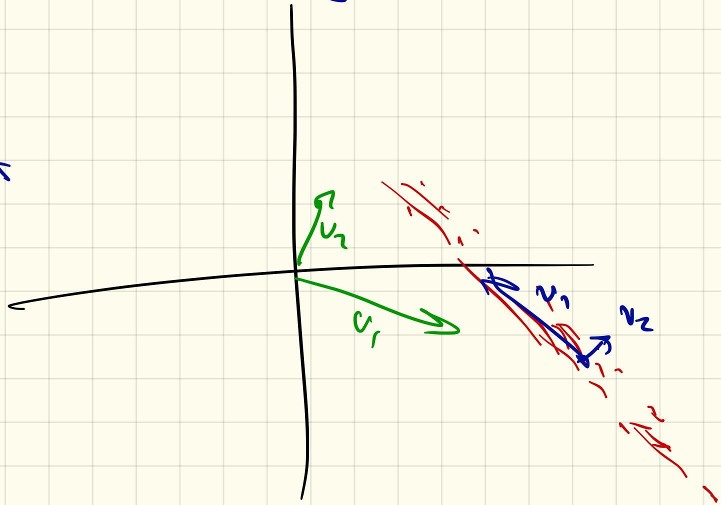
1. Center Data

$A_j \leftarrow$ jth column A

$c_j = \frac{1}{n} A_{ji}$

$\bar{A}_{ji} = A_{ji} - c_j$

2. SVD



Eigenvalues & Eigenvectors

square matrix $B \in \mathbb{R}^{d \times d}$

$$B v = \underbrace{\lambda}_{\text{scalar eigenvalue}} v \qquad \lambda_i = s_i^2$$

$A = USV^T$

d-dim vect

eigenvector

$A \in \mathbb{R}^{n \times d}$

$\hookrightarrow B_R = A^T A \in \mathbb{R}^{d \times d}$ $\qquad B_L = A A^T \in \mathbb{R}^{n \times n}$

$$B_R V = A^T A V = (V S \cancel{U^T})(\cancel{U} S \cancel{V^T})\cancel{V} = V S^2$$

$$B_R U = A A^T U = (U S \cancel{V^T})(\cancel{V} S \cancel{U^T})\cancel{U} = U S^2$$