Degree Reduction for NURBS Symbolic Computation on Curves

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Abstract

Symbolic computation of NURBS plays an important role in many areas of NURBS-based geometric computation and design. However, any nontrivial symbolic computation, especially when rational B-splines are involved, would typically result in B-splines with high degrees. In this paper we develop degree reduction strategies for NURBS symbolic computation on curves. The specific topics we consider include zero curvatures and critical curvatures of plane curves, various ruled surfaces related to space curves, and point/curve bisectors and curve/curve bisectors.

Keywords*: NURBS symbolic computation, degree reduction, zero curvature, critical curvature, torsion, evolute, focal curve, tangent developable, normal scroll, binormal scroll, rectifying developable, bisector curve, bisector surface*

1 Background

Symbolic computation of NURBS [15, 4, 3, 25] refers to algebraic operations on one or more than one NURBS, resulting another NURBS. The operations typically include primitive ones such as sum/difference, multiplication/division[1], differentiation, and composition, and derived ones such as dot product, cross product (of 2 B-splines) and triple-scalar product (of 3 B-splines) in $\mathbb{R}^3$, and generalized cross product (of $n-1$ B-splines) and scalar product (of n B-splines) in $\mathbb{R}^n$, etc. These operations are essential to the construction of complex NURBS models from simple ones, and more importantly to curve/surface interrogation [4, 7, 20, 8, 24]. NURBS symbolic computation has also been used in various other areas such as computing bisectors [9, 10, 11], blending surfaces [18] and offsetting curves [6]. Actually, with the help of rational constraint solvers [27, 12], NURBS symbolic computation plays a fundamental role virtually in every area of geometric computation on free-form curves/surfaces. However, there is one critical problem that severely restricts the power of NURBS symbolic computation - the rapidly increasing degree of the derived NURBS, which is especially true when rational B-splines are involved.

Let us first review briefly symbolic computation on *polynomial* B-splines. To symbolically add/subtract two polynomial B-splines, both degree elevation and knot vector refinement have to be done so that both operands have the same degree and knot vector, and then addition/subtraction is simply applied point-wise to their control polygons. Dividing a polynomial B-spline by another polynomial scalar B-spline basically follows the same procedure, except that the final step is division and the derived B-spline is rational. Differentiation of a polynomial B-spline is simple and has the favorable property of decreasing the degree by one. Multiplication of two B-splines, though, is complicated with different approaches [22, 4, 28]; what is of concern to this paper is that polynomial B-spline multiplication results in another polynomial B-spline with a degree that is the sum of the degrees of the two operands.

In contrast, rational NURBS symbolic computation is a completely different situation. Every operation on *rational* B-spline(s) is derived rather than primitive, and is implemented typically as more than one primitive operations on the denominator(s) and numerator(s). As a result, addition/subtraction/division raises the degree just like multiplication does, and differentiation doubles the degree [2]!

Because any operation on rational(s) either adds together or doubles the degree(s), any nontrivial symbolic computation on *rational* B-splines, especially when higher order differentiation is involved, would quite likely become impractical because of the huge degree of the derived B-

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[1] the division is a closed operation on *rational* B-splines, and it only makes sense when the divisor is a scalar B-spline. Notice that this is completely different from the division operator defined in [15], where symbolic computation on *Bézier polynomials* is investigated, and division of two polynomials results in a quotient and a remainder, both being polynomials.

[2] The rational addition and subtraction can be reduced to the polynomial case if the denominators are the same B-splines.

spline. For example, the derivative of the squared curvature of a quadratic rational curve has a degree of 96 (see Fig. 2)!

Considering that most geometric modeling systems abound with rational quadratic curves/surfaces (e.g., circles and spheres), the rapidly increasing degree can pose a serious problem. A common practice in the CAD community to deal with this, or other similar problems caused by the rational representation, is to approximate the rational curve/surface with a polynomial one. On the one hand, this really does not solve the problem at all, and also an acceptable initial approximation error might be amplified significantly in later stages of the design and modeling process. On the other hand, even starting with a polynomial curve/surface, many (like curvature related) interrogations will turn into a rational one quickly.

In this paper, we develop degree reduction strategies for symbolic computation on NURBS *curves*. Typically we transform the considered B-spline in various ways into one with reduced degree. In some situations, the derived B-spline is different from the initial one, yet gives the exact same solution to the considered problem (like critical curvature inquiry); in other situations, the derived B-spline represents the same geometry, but with a different parameterization (like the ruled bisector surface); in yet other situations, the parameterization is even the same, but some redundant terms are eliminated (like the evolute curve).

Also, a few words are in order about parameterization and the term "rational" or "non-rational". Assuming a curve (called primary curve) has a rational parameterization in s, typically any derived curves (including scalar field on the primary curve) are supposed to be parameterized in s also. We call this natural parameterization. Throughout the paper, when we say f is (non)rational, it should be understood as (non)rational under this natural parameterization, unless explicitly stated otherwise. We will also investigate several 2-dimensional derived surfaces related to curves. In some situations, like the bisector surface between two space curves, we are still able to have a natural parameterization. In other situations, however, a natural parameterization does not make as perfect sense, and we are left with much flexibility in choosing the second parameter and/or the iso-curves. This actually turns out to be a positive factor in the sense that we may choose an appropriate *rational* parameterization or an appropriate rational parameterization with *lower* degree. A trivial example is the tangent developable of a rational space curve $\boldsymbol{x}$. If it is parameterized as $\boldsymbol{x}(s) + \zeta\bar{\boldsymbol{T}}(s)$ ($\bar{\boldsymbol{T}}$ is the unit tangent vector), the developable is not even rational in general; if the parameterization is $\boldsymbol{x}(s) + \eta\boldsymbol{x}'(s)$, the developable is now rational; and finally the same developable can also be represented as $\boldsymbol{x} + \lambda\frac{\mathbf{D}_1}{w(s)} = \frac{\boldsymbol{p}(s)+\lambda\mathbf{D}_1(s)}{w(s)}$ (where $\boldsymbol{x} = \frac{p}{w}, \mathbf{D}_1 = \boldsymbol{p}'w - \boldsymbol{p}w'$; cf. Eq. (2.2)), which has a lower degree[3]. Later in this paper, this re-parameterization strategy is used to represent the rational rectifying developable (Section 4) and the ruled bisector surface (Section 5.2; see also Fig. 4 and Figure 5).

The rest of the paper is organized as follows. Adapted from [1], Section 2 develops the various derivatives of a *rational* B-spline. It shows that the derivatives can actually be expressed in some *polynomial* B-splines up to a *common divisor* and some additive terms involving lower order derivatives; this is a simple yet extremely important observation for NURBS symbolic computation. Section 3 develops degree reduction strategies for two common tasks of curve interrogations, namely finding the zero curvature points and the critical curvature points; also discussed is the derivation of a degree reduced representation of the evolute of the primary curve. In Section 4, several derived B-spline surfaces related to a space curve are investigated. Section 5 develops a polynomial formulation of a linear system defining the curve/curve bisector or point/curve bisector and thus reduces the degree of the bisector that is the solution to the linear system; moreover, a direct approach to solving the ruled point/curve bisector from a 2×3 linear system is proposed therein. Finally, the paper concludes in Section 6.

2 Derivatives of Rational Curves

Suppose $\boldsymbol{x} = \frac{\boldsymbol{p}}{w}$, where p is a polynomial B-spline curve, and w is a polynomial B-spline function. The deriva-

[3]The example just serves as an illustration of the idea; the degree reduction, in this case, is only 1.

tives are,

$$\boldsymbol{x}' = \frac{\boldsymbol{p}'w - \boldsymbol{p}w'}{w^2},$$

$$\begin{aligned}\boldsymbol{x}'' &= \frac{(\boldsymbol{p}'w - \boldsymbol{p}w')'}{w^2} + (\boldsymbol{p}'w - \boldsymbol{p}w')(\frac{1}{w^2})' \\ &= \frac{\boldsymbol{p}''w - \boldsymbol{p}w''}{w^2} + (\boldsymbol{p}'w - \boldsymbol{p}w')(\frac{1}{w^2})'\end{aligned}$$

$$\begin{aligned}\boldsymbol{x}''' = &\frac{\boldsymbol{p}'''w - \boldsymbol{p}w'''}{w^2} + \frac{\boldsymbol{p}''w' - \boldsymbol{p}'w''}{w^2} + \\ &(\boldsymbol{p}''w - \boldsymbol{p}w'')(\frac{1}{w^2})' + \\ &(\boldsymbol{p}''w - \boldsymbol{p}w'')(\frac{1}{w^2})' + (\boldsymbol{p}'w - \boldsymbol{p}w')(\frac{1}{w^2})''\end{aligned}$$

$$\begin{aligned}= &\frac{\boldsymbol{p}'''w - \boldsymbol{p}w'''}{w^2} + \frac{\boldsymbol{p}''w' - \boldsymbol{p}'w''}{w^2} + \\ &2(\boldsymbol{p}''w - \boldsymbol{p}w'')(\frac{1}{w^2})' + \\ &(\boldsymbol{p}'w - \boldsymbol{p}w')(\frac{1}{w^2})''.\end{aligned}$$

By introducing the notations [4],

$$\begin{array}{lll}\mathbf{D}_1 & = & \boldsymbol{p}'w - \boldsymbol{p}w' \\ \mathbf{D}_2 & = & \boldsymbol{p}''w - \boldsymbol{p}w'' \\ \mathbf{D}_3 & = & \boldsymbol{p}'''w - \boldsymbol{p}w''' \\ \mathbf{D}_{21} & = & \boldsymbol{p}''w' - \boldsymbol{p}'w''\end{array} \tag{2.1}$$

the derivatives are rewritten as

$$\begin{array}{ll}\boldsymbol{x}' & = \frac{1}{w^2}\mathbf{D}_1 \\ \boldsymbol{x}'' & = \frac{1}{w^2}\mathbf{D}_2 - \frac{2w'}{w^3}\mathbf{D}_1 \\ \boldsymbol{x}''' & = \frac{1}{w^2}\mathbf{D}_3 + \frac{1}{w^2}\mathbf{D}_{21} - \frac{4w'}{w^3}\mathbf{D}_2 + \frac{6w'^2 - 2ww''}{w^4}\mathbf{D}_1\end{array} \tag{2.2}$$

Remark 1 All $\mathbf{D}$'s with various subscriptions are all *polynomial* B-splines. The most common expression in $\mathbf{D}$'s, with their degrees in B-spline representation, are (cf. Eq. (2.1)),

$\mathbf{D}_1$	$2d-1$
$\mathbf{D}_2$	$2d-2$
$\mathbf{D}_3$	$2d-3$
$[\mathbf{D}_1\ \mathbf{D}_2]$	$4d-3$
$[\mathbf{D}_1\ \mathbf{D}_2\ \mathbf{D}_3]$	$6d-6$

where d is the degree of the considered rational B-spline $\boldsymbol{x}$.

[4] For consistent notations, we should have used $\mathbf{D}_{10}, \mathbf{D}_{20}$, and $\mathbf{D}_{30}$ instead of $\mathbf{D}_1, \mathbf{D}_2$, and $\mathbf{D}_3$, respectively. The simpler notations are adopted due to space consideration and also because that $\mathbf{D}_{21}$ is the only situation where double subscripts are required.

In the above, for any two plane vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $[\boldsymbol{a}\ \boldsymbol{b}]$ denotes the determinant of the matrix consisting of the two column vectors. Geometrically, $[\boldsymbol{a}\ \boldsymbol{b}]$ is the signed area of the parallelogram formed out of the two vectors in the plane, and it is also easy to see that

$$[\boldsymbol{a}\ \boldsymbol{b}] = \boldsymbol{a}_{\mathfrak{r}} \cdot \boldsymbol{b},$$

where, $\boldsymbol{a}_{\mathfrak{r}}$ means the 90 degree counter-clock-wise rotation of $\boldsymbol{a}$. On the other hand, for any three space vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$ is the well-known triple scalar product.

Remark 2 It is proved in [1] that for a space curve $\boldsymbol{x}$, $\mathbf{D}_{21}$, as defined in Eq. (2.1), is spanned by $\mathbf{D}_1$ and $\mathbf{D}_2$. Therefore, by Eq. (2.2), the i-th ($i = 0, 1, 2$) order derivative of a *rational* B-spline $\boldsymbol{x}$ is essentially the *polynomial* $\mathbf{D}_i$ up to the *same* scale factor $\frac{1}{w^2}$, and some additive terms involving lower order derivatives.

3 Symbolic Computation on B-Spline Plane Curves

The inquiry of zero curvature points and critical curvature points (i.e., $\kappa' = 0$) of a plane curve are two fundamental issues in geometric design and modeling. The signed curvature scalar field $[\boldsymbol{x}'\ \boldsymbol{x}'']$ and the derivative of the squared curvature $(\kappa^2)'$ are used in [8], respectively, to perform these two important interrogations. We will transform the first B-spline into a degree reduced one. For the second one, we will present a direct formulation of the critical curvature problem without even squaring the curvature and thus achieve significant degree reduction for the final B-spline. Also discussed in this section is the evolute of a plane curve, which has topological significance for distances to the curve [17], and is intimately related to offset curves [14]. The evolute is first observed to have a rational B-spline representation in [5]. We will eliminate several redundant terms implicitly involved in the original representation of the evolute of a *rational* primary curve and thus transform the evolute into a B-spline of lower degree.

3.1 Curvature Zero Set of a Rational Plane Curve

The curvature of a B-spline plane curve $\boldsymbol{x}$ is [23],

$$\kappa = \frac{[\boldsymbol{x}'\ \boldsymbol{x}'']}{|\boldsymbol{x}'|^3}, \tag{3.1}$$

Except for a special class of Pythagorean-hodograph curves [16], κ, as expressed in Eq. (3.1), is generally not a rational due to the radical in the denominator. The zero set of κ, however, is only related to its numerator (the denominator is always non-zero for a regular curve). If $\boldsymbol{x}$ is a polynomial B-spline plane curve of degree d, $f_0 = [\boldsymbol{x}'\ \boldsymbol{x}'']$

is a polynomial scalar B-spline of degree $2d-3$. If $\boldsymbol{x}$ is instead a rational B-spline, f_0 is a rational B-spline of degree $6d$ because the degree is doubled each time a differentiation is applied to a rational B-spline.

For the rational case, the direct symbolic computation of $f_0 = [\boldsymbol{x}'\ \boldsymbol{x}'']$ involves much wasted effort. By Eq. (2.2), the zero set of f_0 is identical to that of $f_1 = [\mathbf{D}_1\ \mathbf{D}_2]$. By Eq. (2.1),

$$\begin{aligned} f_1 &= [\boldsymbol{p}'w - \boldsymbol{p}w'\ \ \boldsymbol{p}''w - \boldsymbol{p}w''], \\ &= [\boldsymbol{p}'\ \boldsymbol{p}'']w^2 - [\boldsymbol{p}\ \boldsymbol{p}'']w'w - [\boldsymbol{p}'\ \boldsymbol{p}]ww'', \end{aligned} \tag{3.2}$$

where we have also used the fact that $[\boldsymbol{p}\ \boldsymbol{p}] = 0$. Eliminating the common factor w, the original curvature zero set problem is transformed into the zero set of a polynomial B-spline f_2,

$$f_2 = [\boldsymbol{p}'\ \boldsymbol{p}'']w - [\boldsymbol{p}\ \boldsymbol{p}'']w' - [\boldsymbol{p}'\ \boldsymbol{p}]w'',$$

which is of degree $3d-3$, a reduction of $3d+3$ from $6d$ of the brute force approach.

3.2 Curvature Critical Set of a Plane Curve

Because the curvature κ of a B-spline plane curve $\boldsymbol{x}$ is generally not rational, further NURBS symbolic computation can not be applied to the curvature function. A common way to get around of this difficulty and to apply symbolic computation and subdivision strategy to find the critical curvature is to square κ [8] and then take the derivative. For the convenience of discussion, let us call this the squaring approach. The squaring approach works fine, except raising the degree of the final B-spline considerably, and also requiring a post-processing to delete the inflection points introduced by the squaring of κ. Specifically, we start with the curvature function in Eq. (3.1). Taking the square turns it into a rational B-spline function,

$$\kappa^2 = \frac{[\boldsymbol{x}'\boldsymbol{x}'']^2}{|\boldsymbol{x}'|^6}. \tag{3.3}$$

κ^2 is a rational B-spline of degree $6d-6$ if $\boldsymbol{x}$ is a polynomial of degree d, and of degree $(2d+4d)*2+2d*6 = 24d$ if $\boldsymbol{x}$ is a rational of degree d. The critical *and zero* curvature set of $\boldsymbol{x}$ is the roots of the numerator of $(\kappa^2)'$, which, if computed algorithmically, has a degree of $(6d-6)*2 = 12(d-1)$ or $24d*2 = 48d$ if $\boldsymbol{x}$ is polynomial or rational, respectively.

Observing that we are actually interested in κ' rather than κ^2, there is a direct way to find the critical curvature points. Instead of squaring κ and make it representable as a NURBS right away, we take the derivative first and then see what need to be done to transform the final result into a NURBS. By the curvature formula (Eq. (3.1)),

$$\begin{aligned} \kappa' &= \left(\frac{[\boldsymbol{x}'\ \boldsymbol{x}'']}{|\boldsymbol{x}'|^3}\right)' = \frac{[\boldsymbol{x}'\ \boldsymbol{x}''']\ |\boldsymbol{x}'|^3 - |\boldsymbol{x}'|^{3'}\ [\boldsymbol{x}'\ \boldsymbol{x}'']}{|\boldsymbol{x}'|^6} \\ &= \frac{(\boldsymbol{x}'\cdot\boldsymbol{x}')[\boldsymbol{x}'\ \boldsymbol{x}'''] - 3(\boldsymbol{x}'\cdot\boldsymbol{x}'')[\boldsymbol{x}'\ \boldsymbol{x}'']}{|\boldsymbol{x}'|^5}. \end{aligned}$$

Denoting the numerator as g_0,

$$g_0 = (\boldsymbol{x}'\cdot\boldsymbol{x}')[\boldsymbol{x}'\ \boldsymbol{x}'''] - 3(\boldsymbol{x}'\cdot\boldsymbol{x}'')[\boldsymbol{x}'\ \boldsymbol{x}''], \tag{3.4}$$

we have that the curvature critical set is identified with the zero set of g_0, an expression given earlier in [2] in coordinate functions of the original plane curve [5]. g_0 can be symbolically computed as a polynomial (rational) B-spline if the curve $\boldsymbol{x}$ is a polynomial (rational) B-spline. If $\boldsymbol{x}$ is a polynomial B-spline of degree d, g_0 has a degree of $4d-6$. For a cubic polynomial curve, the final B-spline degree is only 6, compared to 24, had the squaring approach been taken[6]. See Figure 1[7] for detail.

Now let us focus on the rational situation. Because $\boldsymbol{x}', \boldsymbol{x}''$ and $\boldsymbol{x}'''$ have degrees of $2d, 4d$ and $8d$, respectively, the first and second terms in Eq. (3.4) have degrees of $2d*2+(2d+8d) = 14d$ and $(2d+4d)*2 = 12d$, respectively. Therefore, g_0 has a degree of $14d+12d = 26d$. This is already a significant degree reduction of $48d-26d = 22d$ from the squaring approach. However, more reduction is possible. Substituting Eq. (2.2) into Eq. (3.4), and eliminating the common factor $\frac{1}{w^4}$ transforms g_0 into g_1 with the same zero set,

$$\begin{aligned} g_1 &= [\boldsymbol{x}'\ \boldsymbol{x}''']|\boldsymbol{x}'|^2 - 3(\boldsymbol{x}'\cdot\boldsymbol{x}'')[\boldsymbol{x}'\ \boldsymbol{x}''] \\ &= [\mathbf{D}_1\ \ \mathbf{D}_3]|\mathbf{D}_1|^2 + [\mathbf{D}_1\ \ \mathbf{D}_{21}]|\mathbf{D}_1|^2 - \frac{4w'}{w}[\mathbf{D}_1\ \ \mathbf{D}_2]|\mathbf{D}_1|^2 \\ &\quad - 3\Big((\mathbf{D}_1\cdot\mathbf{D}_2)[\mathbf{D}_1\ \ \mathbf{D}_2] - \frac{2w'}{w}|\mathbf{D}_1|^2[\mathbf{D}_1\ \ \mathbf{D}_2]\Big) \\ &= [\mathbf{D}_1\ \ \mathbf{D}_3]|\mathbf{D}_1|^2 + [\mathbf{D}_1\ \ \mathbf{D}_{21}]|\mathbf{D}_1|^2 + \frac{2w'}{w}[\mathbf{D}_1\ \ \mathbf{D}_2]|\mathbf{D}_1|^2 \\ &\quad - 3(\mathbf{D}_1\cdot\mathbf{D}_2)[\mathbf{D}_1\ \ \mathbf{D}_2] \end{aligned}$$

Multiplying both sides by w does not change the zero set of g_1, and transforms g_1 into a polynomial scalar B-spline g_2,

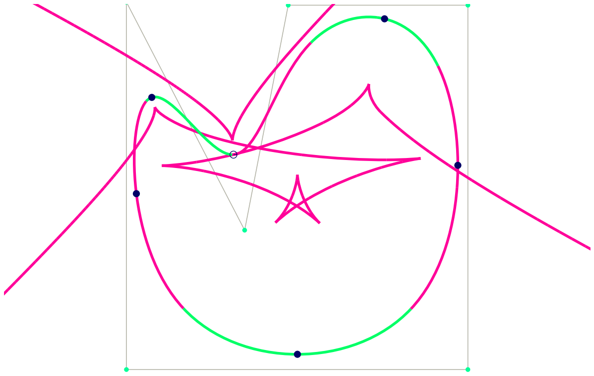
$$\begin{aligned} g_2 = {} & [\mathbf{D}_1\ \ \mathbf{D}_3]|\mathbf{D}_1|^2 w + [\mathbf{D}_1\ \ \mathbf{D}_{21}]|\mathbf{D}_1|^2 w + \\ & 2w'[\mathbf{D}_1\ \ \mathbf{D}_2]|\mathbf{D}_1|^2 - 3(\mathbf{D}_1\cdot\mathbf{D}_2)[\mathbf{D}_1\ \ \mathbf{D}_2]w \end{aligned} \tag{3.5}$$

By Remark 1, all four terms in the RHS of the above equation have the same degree of $9d-6$. Therefore, we finally have a transformed polynomial B-spline f with degree $9d-6$, which is a huge reduction from $48d$ of the brute

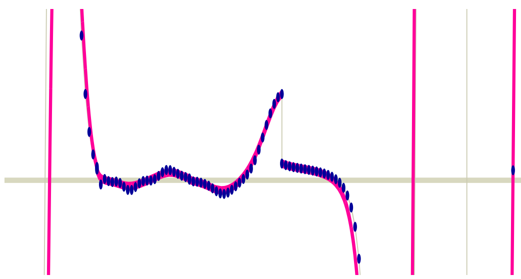
[5] Thank one of our reviewers for directing us to the relevant reference.

[6] This degree comparison, as well as the one on rational curve case discussed shortly, is also tested with Irit version 9.5 [5], where κ^2 and $(\kappa^2)'$ have degrees of 12 and 24 respectively for a cubic polynomial, and 72 and 144 respectively for a cubic rational.

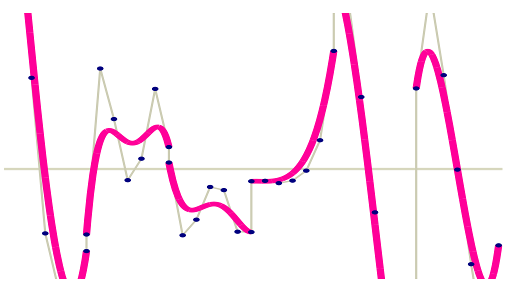
[7] The curve model is from [21].



(a) Curvature Critical Points



(b) Squaring Approach



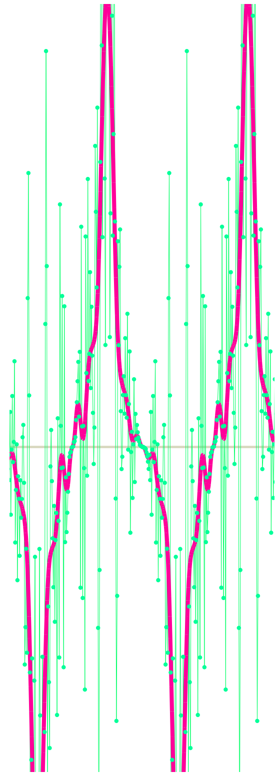
(c) Direct Approach

Figure 1. Symbolically Computing Curvature Critical Points (*polynomial case*)

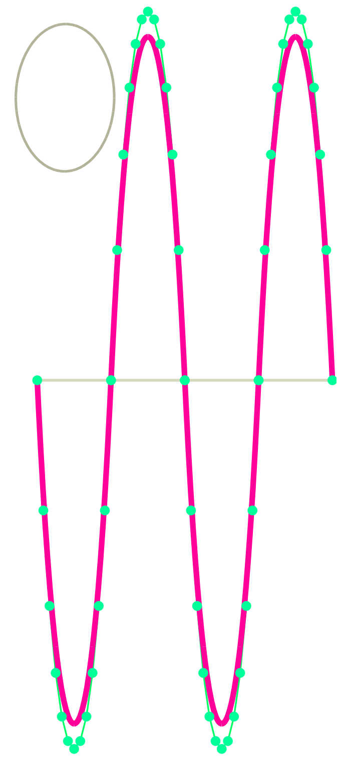
(a) *a cubic B-spline curve* $\boldsymbol{x}$*, its evolute and six* $\kappa' = 0$ *points*[a].
(b) *the graph of the numerator of* $(\kappa^2)'$*, with degree* 24*, and* 150 *control points. Two extra roots are returned where* $\kappa = 0$*.*
(c) *the graph of* $g_0 = [\boldsymbol{x}'\ \boldsymbol{x}'''] |\boldsymbol{x}'|^2 - 3(\boldsymbol{x}' \cdot \boldsymbol{x}'')[\boldsymbol{x}'\ \boldsymbol{x}'']$*, with degree* 6 *and* 24 *control points.* f *has exactly* 6 *roots, where* $\kappa' = 0$*.*

All control polygons are shown in light gray. Notice that (b) and (c) are non-uniformly scaled and also trimmed to fit the space.

[a] There are actually 8 *extreme* curvature points (corresponding to 8 evolute cusps), with the 2 extra ones (not shown on the primary curve) at the 2 ends of the bottom segment (segments are in alternating colors), where the left and right limit κ' have different sign.



(a) Squaring Approach



(b) Direct Approach

Figure 2. Symbolically Computing Curvature Critical Points (*rational case*)

(a) *the graph (top and bottom trimmed off to fit the space) of the numerator of* $(\kappa^2)'$ *of an ellipse* $\boldsymbol{x}$*, with degree* 96*,* 388 *control points. Vertically scaled down by* 0.00001*.*

(b) *the ellipse* $\boldsymbol{x}$*, and the graph of* g_2 *in Eq. (3.5), with degree* 12 *and* 52 *control points. Vertically scaled down by* 0.3*.*

Control polygons are shown in green in both images. Notice that there are actually only 4 *roots for each graph, because the rightmost one identifies with the leftmost one by periodic condition. Also notice that both graphs are smooth but with* C^{-1} *B-spline representation.*

force squaring approach. For a cubic rational curve, the final B-spline is only degree 21, compared to 144, had the squaring approach been taken. Fig. 2 shows the comparison on finding the 4 critical curvature points of an ellipse represented as a C^0 quadratic rational B-spline.

3.3 The Evolute of a Plane Curve

The evolute of a plane curve $\boldsymbol{x}$ is [26, 19],

$$\boldsymbol{\mathcal{E}}(x) = \boldsymbol{x} + \frac{1}{\kappa}\bar{\boldsymbol{N}},$$

where $\kappa = \frac{|\boldsymbol{x}'|^3}{[\boldsymbol{x}' \ \boldsymbol{x}'']}$ is the *signed* curvature w.r.t. the unit normal $\bar{\boldsymbol{N}} = \frac{\boldsymbol{x}'_{\mathfrak{r}}}{|\boldsymbol{x}'_{\mathfrak{r}}|}$ [8]. It can be transformed into,

$$\boldsymbol{\mathcal{E}}(x) = \boldsymbol{x} + \frac{|\boldsymbol{x}'|^3}{[\boldsymbol{x}' \ \boldsymbol{x}'']}\frac{\boldsymbol{x}'_{\mathfrak{r}}}{|\boldsymbol{x}'_{\mathfrak{r}}|} = \boldsymbol{x} + \frac{|\boldsymbol{x}'|^2}{[\boldsymbol{x}' \ \boldsymbol{x}'']}\boldsymbol{x}'_{\mathfrak{r}}, \qquad (3.6)$$

which is rational of degree $3d-3$ if $\boldsymbol{x}$ is a polynomial of degree d, and of degree $13d$ if $\boldsymbol{x}$ is a rational of degree d.

For the rational case, based on the definition of $\mathbf{D}$'s (Eq. (2.1)), and its relation to the derivatives (Eq. (2.2)), we are again able to transform $\boldsymbol{\mathcal{E}}(x)$, achieving much degree reduction. Specifically, substituting Eq. (2.2), and

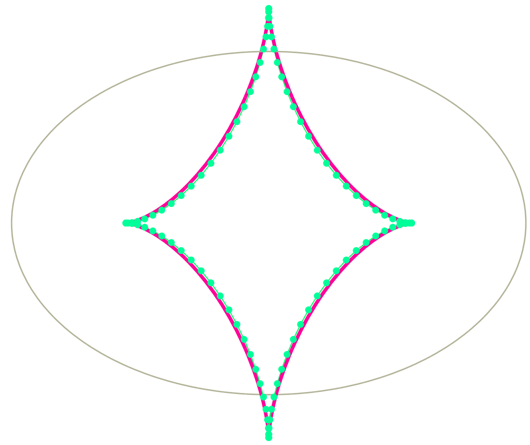
$$\boldsymbol{x}'_{\mathfrak{r}} = \left(\frac{\mathbf{D}_1}{w^2}\right)_{\mathfrak{r}} = \frac{\mathbf{D}_{1\mathfrak{r}}}{w^2},$$

into Eq. (3.6), the evolute of a rational plane curve is reformulated as,

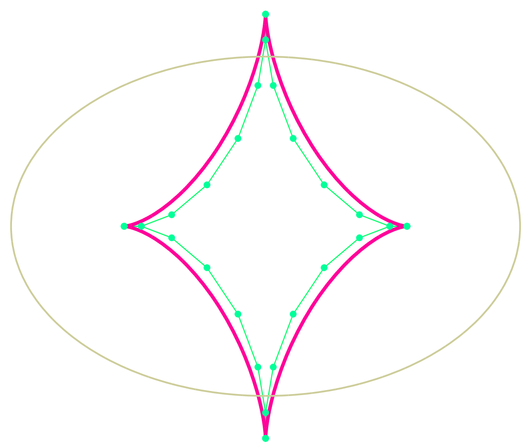
$$\boldsymbol{\mathcal{E}}(x) = \frac{\boldsymbol{p}}{w} + \frac{|\mathbf{D}_1|^2}{[\mathbf{D}_1 \ \mathbf{D}_2]}\frac{\mathbf{D}_{1\mathfrak{r}}}{w^2} = \frac{[\mathbf{D}_1 \ \mathbf{D}_2]w\,\boldsymbol{p} + |\mathbf{D}_1|^2\,\mathbf{D}_{1\mathfrak{r}}}{[\mathbf{D}_1 \ \mathbf{D}_2]w^2},$$

with a final degree of $6d-3$ (cf. Remark 1). Fig. 3 shows the evolute of an ellipse, comparing initial representation to the transformed and degree reduced one.

Notice that we are able to reduce the degree considerably because (cf. Remark 2) the $\boldsymbol{x}^{(i)}$'s can be replaced with the $\mathbf{D}_i$'s and the algebraic operations on the common divisor w^2 result in some high degree terms that can be canceled out.



(a) $\boldsymbol{\mathcal{E}}(x) = \boldsymbol{x} + \frac{|\boldsymbol{x}'|^2}{[\boldsymbol{x}' \ \boldsymbol{x}'']}\boldsymbol{x}'_{\mathfrak{r}}$, degree = 26, # ctrl pnts = 108



(b) $\boldsymbol{\mathcal{E}}(x) = \boldsymbol{x} + \frac{|\mathbf{D}_1|^2}{[\mathbf{D}_1 \ \mathbf{D}_2]}\boldsymbol{x}'_{\mathfrak{r}}$, degree = 9, # ctrl pnts = 40

Figure 3. Evolute of an Ellipse

4 Symbolic Computation on Space B-Spline Curves

For a space curve, a local orthogonal basis is,

$$\begin{cases} \boldsymbol{T} &= \boldsymbol{x}' \\ \boldsymbol{B} &= \boldsymbol{x}' \times \boldsymbol{x}'' \\ \boldsymbol{N} &= \boldsymbol{B} \times \boldsymbol{x}' = (\boldsymbol{x}' \times \boldsymbol{x}'') \times \boldsymbol{x}' \end{cases} \qquad (4.1)$$

The normalized one, called the Frenet frame, is,

$$\begin{cases} \bar{\boldsymbol{T}} &= \frac{\boldsymbol{x}'}{|\boldsymbol{x}'|} \\ \bar{\boldsymbol{B}} &= \frac{\boldsymbol{x}' \times \boldsymbol{x}''}{|\boldsymbol{x}' \times \boldsymbol{x}''|} \\ \bar{\boldsymbol{N}} &= \frac{\boldsymbol{N}}{|\boldsymbol{N}|} = \frac{(\boldsymbol{x}' \times \boldsymbol{x}'') \times \boldsymbol{x}'}{|\boldsymbol{x}' \times \boldsymbol{x}''|\ |\boldsymbol{x}'|} \end{cases} \qquad (4.2)$$

[8] The same evolute is defined if the Frenet-frame normal (and consequently always positive κ) had been used.

Curvature We will not go into the details on curvature zero set and critical set computation. Instead, we would like to point out that although κ is not rational in general, both $\kappa\bar{\boldsymbol{N}}$ and $\frac{1}{\kappa}\bar{\boldsymbol{N}}$ are rational just as in the plane curve case. Recall that the curvature of a space curve is,

$$\kappa = \frac{|\boldsymbol{x}' \times \boldsymbol{x}''|}{|\boldsymbol{x}'|^3} \qquad (4.3)$$

and thus,

$$\begin{aligned}\kappa\bar{\boldsymbol{N}} &= \frac{|\boldsymbol{x}'\times\boldsymbol{x}''|}{|\boldsymbol{x}'|^3}\,\bar{\boldsymbol{N}}\\ &= \frac{|\boldsymbol{x}'\times\boldsymbol{x}''|}{|\boldsymbol{x}'|^3}\,\frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\times\boldsymbol{x}'}{|\boldsymbol{x}'\times\boldsymbol{x}''|\,|\boldsymbol{x}'|}\\ &= \frac{1}{|\boldsymbol{x}'|^4}\,(\boldsymbol{x}'\times\boldsymbol{x}'')\times\boldsymbol{x}',\end{aligned}$$

and,

$$\begin{aligned}\frac{1}{\kappa}\bar{\boldsymbol{N}} &= \frac{|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\,\bar{\boldsymbol{N}}\\ &= \frac{|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\,\frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\times\boldsymbol{x}'}{|\boldsymbol{x}'\times\boldsymbol{x}''|\,|\boldsymbol{x}'|}\\ &= \frac{|\boldsymbol{x}'|^2}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}\,(\boldsymbol{x}'\times\boldsymbol{x}'')\times\boldsymbol{x}';\end{aligned} \tag{4.4}$$

that is, both $\kappa\bar{\boldsymbol{N}}$ and $\frac{1}{\kappa}\bar{\boldsymbol{N}}$ are rationals.

Torsion The torsion of a space curve is [23],

$$\tau = \frac{[\boldsymbol{x}'\boldsymbol{x}''\boldsymbol{x}''']}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}, \tag{4.5}$$

and obviously it is a rational. If $\boldsymbol{x}$ is a rational of degree d, τ, as defined above, is symbolically computed to be a rational B-spline with degree $(2d+4d+8d)+(2d+4d)*2 = 26d$. By Eq. (2.2) and Remark 2,

$$\tau = w^2\frac{[\mathbf{D}_1\,\mathbf{D}_2\,\mathbf{D}_3]}{|\mathbf{D}_1\times\mathbf{D}_2|^2}, \tag{4.6}$$

which, by Remark 1, is a rational scalar B-spline with degree $8d-6$. Moreover, the zero set of τ is identical to that of the polynomial scalar B-spline,

$$f = [\mathbf{D}_1\,\mathbf{D}_2\,\mathbf{D}_3],$$

which has a degree of $6d-6$, compared to a degree of $14d$ if computed directly from $[\boldsymbol{x}'\boldsymbol{x}''\boldsymbol{x}''']$.

Tangent Developable $\mathcal{D}_t$, Normal Scroll $\mathcal{D}_n$ and Binormal Scroll $\mathcal{D}_b$, are all ruled surfaces, and are generated on the same directrix $\boldsymbol{x}$ with generator vectors $\bar{\boldsymbol{T}}$, $\bar{\boldsymbol{N}}$ and $\bar{\boldsymbol{B}}$, respectively. The generators can be replaced by their unnormalized counterparts in Eq. (4.1), and consequently we have

$$\begin{aligned}\mathcal{D}_t &= \boldsymbol{x} + \zeta\boldsymbol{x}',\\ \mathcal{S}_b &= \boldsymbol{x} + \zeta(\boldsymbol{x}'\times\boldsymbol{x}''),\\ \mathcal{S}_n &= \boldsymbol{x} + \zeta(\boldsymbol{x}'\times\boldsymbol{x}'')\times\boldsymbol{x}',\end{aligned}$$

where ζ is the parameter along the various ruled directions. If $\boldsymbol{x}$ is rational of degree d, these ruled surfaces have degree $3d$, $7d$ and $9d$, respectively. However, the above generators can be replaced once again by their counterparts of $\mathbf{D}$'s (cf. Eq. (2.1)); specifically, after some derivation (omitted here), these ruled surfaces are re-parameterized (in λ) as,

$$\begin{aligned}\mathcal{D}_t &= \frac{p+\lambda\mathbf{D}_1}{w},\\ \mathcal{S}_b &= \frac{p+\lambda(\mathbf{D}_1\times\mathbf{D}_2)}{w},\\ \mathcal{S}_n &= \frac{p+\lambda(\mathbf{D}_1\times\mathbf{D}_2)\times\mathbf{D}_1}{w},\end{aligned}$$

with degrees $2d-1$, $4d-3$ and $6d-4$, respectively.

The Rectifying Developable $\mathcal{D}_r$ is the developable surface developed, on the directrix $\boldsymbol{x}$, by the Darboux vector [19] $\boldsymbol{D} = \tau\bar{\boldsymbol{T}} + \kappa\bar{\boldsymbol{B}}$; that is,

$$\mathcal{D}_r = \boldsymbol{x} + \zeta(\tau\bar{\boldsymbol{T}} + \kappa\bar{\boldsymbol{B}}),$$

where ζ is the parameter of the rectifying developable surface on the ruled direction. Of all the terms in the RHS of the above equation, $\bar{\boldsymbol{T}}$, $\bar{\boldsymbol{B}}$ and κ are not rational in general. However, by Eq. (4.2) (4.5) and (4.3),

$$\begin{aligned}\tau\bar{\boldsymbol{T}} + \kappa\bar{\boldsymbol{B}} &= \frac{[\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}''']}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}\,\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|} + \frac{|\boldsymbol{x}'\times\boldsymbol{x}''|}{|\boldsymbol{x}'|^3}\,\frac{\boldsymbol{x}'\times\boldsymbol{x}''}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\\ &= \frac{1}{|\boldsymbol{x}'|}\Big(\frac{[\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}''']}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}\,\boldsymbol{x}' + \frac{\boldsymbol{x}'\times\boldsymbol{x}''}{|\boldsymbol{x}'|^2}\Big).\end{aligned}$$

Hence, by a simple re-parameterization of $\lambda = |\boldsymbol{x}'|\,\zeta$, $\mathcal{D}_r$ is actually a rational,

$$\mathcal{D}_r = \boldsymbol{x} + \lambda\Big(\frac{[\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}''']}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}\,\boldsymbol{x}' + \frac{\boldsymbol{x}'\times\boldsymbol{x}''}{|\boldsymbol{x}'|^2}\Big),$$

with a degree of $39d$ assuming $\boldsymbol{x}$ is a rational of degree d. Similarly, replacing the derivatives with $\mathbf{D}$'s (Eq. (2.2)), $\mathcal{D}_r$ is transformed into a rational

$$\mathcal{D}_r = \boldsymbol{x} + \lambda\Big(\frac{[\mathbf{D}_1\,\mathbf{D}_2\,\mathbf{D}_3]}{|\mathbf{D}_1\times\mathbf{D}_2|^2}\,\mathbf{D}_1 + \frac{\mathbf{D}_1\times\mathbf{D}_2}{|\mathbf{D}_1|^2}\Big),$$

with a degree of $13d-8$

The Focal Curve is not Rational in General. For a plane curve, the locus of its curvature centers or osculating circle centers, i.e. the evolute, has significant topological meaning in various applications. On the other hand, there are two similar curves related to a *space* curve - they are the locus of its osculating *circle* centers and the locus of its osculating *sphere* centers. In this paper, we call them evolute and focal curve, respectively. The evolute of a space curve turns out to be a rational, just like its counterpart in plane curve case (See Section 3.3); however it does not have as much topological significance as the focal curve does. Therefore, instead of transforming and reducing the degree of the evolute, we will work on the focal curve. The work, though, is of a negative type - we will show that, unfortunately, the focal curve of a space curve is non-rational, in general.

The focal curve of a space curve x is the locus of osculating sphere centers, or,

$$\boldsymbol{\mathcal{F}}_x = \boldsymbol{x} + \frac{1}{\kappa}\bar{\boldsymbol{N}} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}}. \tag{4.7}$$

The first term is rational, and by Eq.(4.4), the second term is rational too. However, $\boldsymbol{\mathcal{F}}_x$ is not rational in general, because the third term of the RHS of Eq. (4.7) is not rational as proved below.

By Eq. (3.1), Eq. (4.5) and Eq. (4.2),

$$\begin{aligned}\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}} &= \left(\frac{|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\right)'\frac{\boldsymbol{x}'\times\boldsymbol{x}''}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\\ &= \frac{3|\boldsymbol{x}'|^2\,|\boldsymbol{x}'|'\,|\boldsymbol{x}'\times\boldsymbol{x}''| - |\boldsymbol{x}'\times\boldsymbol{x}''|'\,|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}\frac{\boldsymbol{x}'\times\boldsymbol{x}''}{|\boldsymbol{x}'\times\boldsymbol{x}''|}.\end{aligned}$$

Observing that

$$\begin{aligned}|\boldsymbol{x}'\times\boldsymbol{x}''|' &= \left(\sqrt{(\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}'')}\right)'\\ &= \frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}'')'}{|\boldsymbol{x}'\times\boldsymbol{x}''|}\\ &= \frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}''')}{|\boldsymbol{x}'\times\boldsymbol{x}''|},\end{aligned}$$

this is,

$$\begin{aligned}&\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}}\\ &= \frac{3|\boldsymbol{x}'|^2\,|\boldsymbol{x}'|' - \frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}''')}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}(\boldsymbol{x}'\times\boldsymbol{x}'')\\ &= \frac{3|\boldsymbol{x}'|(\boldsymbol{x}'\cdot\boldsymbol{x}'') - \frac{(\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}''')}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}|\boldsymbol{x}'|^3}{|\boldsymbol{x}'\times\boldsymbol{x}''|^2}(\boldsymbol{x}'\times\boldsymbol{x}'')\\ &= \frac{|\boldsymbol{x}'|}{|\boldsymbol{x}'\times\boldsymbol{x}''|^4}(\boldsymbol{x}'\times\boldsymbol{x}'')\\ &\quad\left(3(\boldsymbol{x}'\cdot\boldsymbol{x}'')|\boldsymbol{x}'\times\boldsymbol{x}''|^2 - (\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}''')|\boldsymbol{x}'|^2\right),\end{aligned}$$

where we have already used the fact,

$$|\boldsymbol{x}'||\boldsymbol{x}'|' = \frac{1}{2}(|\boldsymbol{x}'|^2)' = \frac{1}{2}(\boldsymbol{x}'\cdot\boldsymbol{x}')' = \boldsymbol{x}'\cdot\boldsymbol{x}''.$$

By introducing

$$\Psi = \frac{3(\boldsymbol{x}'\cdot\boldsymbol{x}'')|\boldsymbol{x}'\times\boldsymbol{x}''|^2 - (\boldsymbol{x}'\times\boldsymbol{x}'')\cdot(\boldsymbol{x}'\times\boldsymbol{x}''')|\boldsymbol{x}'|^2}{|\boldsymbol{x}'\times\boldsymbol{x}''|^4}(\boldsymbol{x}'\times\boldsymbol{x}''),$$

we have,

$$\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}} = |\boldsymbol{x}'|\Psi.$$

Ψ is a rational because each of its term is. If $\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}}$ is a rational, then solving the above equation for $|\boldsymbol{x}'|$, it would also be rational. Henceforth, $\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}}$ can not be rational, provided that $|\boldsymbol{x}'|$ is not rational, which is generally true.

The Polar Developable or the Focal Surface $\boldsymbol{\mathcal{D}}_p$ is developed by the focal lines (lines parallel to binormals and passing through the curvature centers) on the focal curve, i.e.,

$$\boldsymbol{\mathcal{D}}_p = \boldsymbol{x} + \frac{1}{\kappa}\bar{\boldsymbol{N}} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}} + \lambda\,\bar{\boldsymbol{B}},$$

where λ is the parameter of the polar developable surface on the ruled direction. This is not a rational parameterization. However, rewriting (i.e., re-parameterizing) the last two terms as,

$$\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\bar{\boldsymbol{B}} + \lambda\,\bar{\boldsymbol{B}} = \mu\boldsymbol{B},$$

for some $\mu \in \mathbb{R}$, the polar developable now has a rational parameterization,

$$\boldsymbol{\mathcal{D}}_p = \boldsymbol{x} + \frac{1}{\kappa}\bar{\boldsymbol{N}} + \mu\boldsymbol{B}, \tag{4.8}$$

because the second term is already shown to be rational (cf. Eq. (4.4)), and $B = \boldsymbol{x'}\times\boldsymbol{x''}$ is rational too. By Eq. (4.4),

$$\boldsymbol{\mathcal{D}}_p = \boldsymbol{x} + \frac{|\boldsymbol{x'}|^2}{|\boldsymbol{x'}\times\boldsymbol{x''}|^2}(\boldsymbol{x'}\times\boldsymbol{x''})\times\boldsymbol{x'} + \mu\,\boldsymbol{x'}\times\boldsymbol{x''}. \tag{4.9}$$

If $\boldsymbol{x}$ is a rational B-spline, again by Eq. (2.2), $\boldsymbol{\mathcal{D}}_p$ is transformed into,

$$\boldsymbol{\mathcal{D}}_p = \frac{p}{w} + \frac{1}{w^2}\frac{|\mathbf{D}_1|^2}{|\mathbf{D}_1\times\mathbf{D}_2|^2}(\mathbf{D}_1\times\mathbf{D}_2)\times\mathbf{D}_1 + \frac{\mu}{w^4}\mathbf{D}_1\times\mathbf{D}_2$$

with significantly reduced degree.

5 Point/Curve and Curve/Curve Bisectors

It has been proved [13, 10] that the bisector[9] between a rational plane curve and a plane point is a rational curve, that between a space point and a space rational curve is a

[9] Strictly speaking, a point on the bisector, as discussed in this section and in [10], is not *equidistant* to the point and to the curve (or to the two curves), even locally – rather it has equal critical distances. To get the *real* bisector, trimming procedure as done in [13] for the plane case has to be applied.

rational ruled surface, and that between two space rational curves is a rational surface. For the plane case, [13] has shown that the bisector has a maximal degree of $3d-1$ and $4d-1$ if the curve is a degree d polynomial and rational respectively[10] . We will reformulate bisectors on space curves [10] to achieve precisely the same result.

5.1 Polynomial Formulation of the Linear System for Solving the Curve/Curve Bisector

Adapted from [10], the bisector $\boldsymbol{\mathcal{B}}$ of two space curves $\boldsymbol{x}(s)$ and $\hat{\boldsymbol{x}}(\hat{s})$ is the solution to the following linear system,

$$\begin{pmatrix} \boldsymbol{x}' \\ \hat{\boldsymbol{x}}' \\ \boldsymbol{x}-\hat{\boldsymbol{x}} \end{pmatrix} \boldsymbol{\mathcal{B}} = \begin{pmatrix} \boldsymbol{x}\cdot\boldsymbol{x}' \\ \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}' \\ \frac{|\boldsymbol{x}|^2-\hat{\boldsymbol{x}}^2}{2} \end{pmatrix}, \tag{5.1}$$

where each B-spline in the LHS is regarded as a row vector. Geometrically, this simply means that a bisector point $\boldsymbol{\mathcal{B}}(s,\hat{s})$ of two space curves $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ is the intersection point of three planes namely, the plane passing $\boldsymbol{x}(s)$ with normal $\boldsymbol{x}'(s)$, the plane passing $\hat{\boldsymbol{x}}(\hat{s})$ with normal $\hat{\boldsymbol{x}}'(\hat{s})$, and the plane passing $\frac{\boldsymbol{x}(s)+\hat{\boldsymbol{x}}(\hat{s})}{2}$ with normal $\boldsymbol{x}(s)-\hat{\boldsymbol{x}}(\hat{s})$. By Cramer's rule, the bisector is solved as,

$$\boldsymbol{\mathcal{B}} = \frac{(\mathbb{D}_0,\mathbb{D}_1,\mathbb{D}_2)}{\mathbb{D}}, \tag{5.2}$$

where $\mathbb{D}$ is the determinant of the LHS matrix, and $\mathbb{D}_i$ ($i=0,1,2$) is that of the LHS matrix when its i-th column is replaced by the RHS column vector.

Assume that $\boldsymbol{x}$ is a degree d B-spline, either polynomial or rational, with parameter s. Only focusing on degree in s ($\boldsymbol{\mathcal{B}}$ has the same degree in $\hat{s}$ as that in s), Eq. (5.1) can be re-written as,

$$\begin{pmatrix} d-1 & d-1 & d-1 \\ 0 & 0 & 0 \\ d & d & d \end{pmatrix} \boldsymbol{\mathcal{B}} = \begin{pmatrix} 2d-1 \\ 0 \\ 2d \end{pmatrix}. \tag{5.3}$$

First consider the situation when $\boldsymbol{x}$ is a degree d *polynomial*. $\mathbb{D}$, as a 3×3 determinant, is the sum of 6 terms. Because each of the 6 terms is a degree $2d-1$ polynomial B-spline, $\mathbb{D}$ is also a degree $2d-1$ polynomial B-spline. On the other hand, $\mathbb{D}_0$ has the following degree representation,

$$\begin{pmatrix} 2d-1 & d-1 & d-1 \\ 0 & 0 & 0 \\ 2d & d & d \end{pmatrix} \tag{5.4}$$

and thus 4 of the 6 terms are degree $3d-1$ B-splines, while the other 2 are still degree $2d-1$ polynomial B-spline. Therefore, $\mathbb{D}_0$, and of course $\mathbb{D}_1$ and $\mathbb{D}_2$ as well, is a degree $3d-1$ polynomial B-spline. Finally, a division of a degree $3d-1$ polynomial Bspline by another $2d-1$ polynomial scalar B-spline results in $\boldsymbol{\mathcal{B}}$ being a rational B-spline of degree $3d-1$.

Now consider the situation when $\boldsymbol{x}$ is rational. $\mathbb{D}$ is again the summation of 6 terms, each of which is a degree $2d-1$ *rational* B-splines. However, adding them together does not raise degree at all, because all the denominators of these 6 rational B-splines are the same and the summation of the 6 rationals is reduced to the summation of 6 *polynomial* B-splines followed by a division by the common scalar *polynomial* B-spline. Therefore, $\mathbb{D}$ is finally a degree $2d-1$ rational B-spline. On the other hand, $\mathbb{D}_0$ (and similarly for $\mathbb{D}_1$ and $\mathbb{D}_2$) is a different situation. Notice that each of the three rational scalar B-splines in the first column of Eq. (5.4) has a different denominator from that of the other two in the corresponding row. That is to say, in the summation of the determinant, 2 of the 6 rational B-spline terms have the common denominator, yet another 2 have another common denominator, and the rest 2 have yet another common denominator. Therefore, applying addition first within each pair and then across the 3 pairs, the final rational B-spline is symbolically computed to have a degree of $(3d-1)+(3d-1)+(2d-1)=8d-3$. A much better and also simpler way is to bring the three rational B-splines in each row of $\mathbb{D}_0$ to the same type by degree elevation and knot insertion, i.e. transforming Eq. (5.4) into,

$$\begin{pmatrix} 2d-1 & 2d-1 & 2d-1 \\ 0 & 0 & 0 \\ 2d & 2d & 2d \end{pmatrix}. \tag{5.5}$$

Now adding the 6 terms in the determinant as usual, $\mathbb{D}_0$ is symbolically computed to be a degree $(2d-1)+2d=4d-1$ rational B-spline. And finally, $\boldsymbol{\mathcal{B}}$ (cf. Eq. (5.2)) is a rational B-spline of degree $(2d-1)+(4d-1)=6d-2$.

What we have just discussed is all about the optimal implementation. In what follows, we will show that, for the rational case, the degree actually can be reduced to $4d-1$ by some careful mathematical reformulation.

Suppose $\boldsymbol{x}=\frac{\boldsymbol{p}}{w}$. Noticing that only the directions of the plane normals matter, two of the three plane normals, $\boldsymbol{x}'$ and $\boldsymbol{x}-\hat{\boldsymbol{x}}$, can be replaced with $\mathbf{D}_1$ and $\boldsymbol{p}-w\hat{\boldsymbol{x}}$, respectively. Consequently, Eq. (5.1) can be transformed into,

$$\begin{pmatrix} \mathbf{D}_1 \\ \hat{\boldsymbol{x}}' \\ \boldsymbol{p}-\hat{\boldsymbol{x}}w \end{pmatrix} \bar{\boldsymbol{\mathcal{B}}} = \begin{pmatrix} \boldsymbol{p}\cdot\mathbf{D}_1, \\ w\,\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}', \\ \frac{\boldsymbol{p}^2-(\hat{\boldsymbol{x}}w)^2}{2}, \end{pmatrix}, \tag{5.6}$$

where,

$$\bar{\boldsymbol{\mathcal{B}}} = w\boldsymbol{\mathcal{B}}. \tag{5.7}$$

[10] It is stated to be $4d-2$ in [13] for the rational case. We suspect that it is just a typo.

Only focusing on degree in s, this is,

$$\begin{pmatrix} 2d-1 & 2d-1 & 2d-1 \\ 0 & 0 & 0 \\ d & d & d \end{pmatrix} \bar{\boldsymbol{\mathcal{B}}} = \begin{pmatrix} 3d-1 \\ d \\ 2d \end{pmatrix}. \quad (5.8)$$

Therefore, we are back to the polynomial situation; and analogously, $\mathbb{D}$ is a degree $3d-1$ polynomial B-spline, and $\mathbb{D}_i$ $(i = 0, 1, 2)$ is a degree $4d-1$ polynomial B-spline. Noticing that now $\boldsymbol{\mathcal{B}} = \frac{\bar{\boldsymbol{\mathcal{B}}}}{w} = \frac{(\mathbb{D}_0, \mathbb{D}_1, \mathbb{D}_2)}{\mathbb{D}w}$ (cf.Eq.(5.7)), and that $\mathbb{D}w$ is a degree $4d-1$ polynomial B-spline, $\boldsymbol{\mathcal{B}}$ is finally symbolically computed as a rational B-spline of degree $4d-1$ in s.

5.2 Ruled Point/Curve Bisectors

If one of the curve, say $\hat{\boldsymbol{x}}$, is degenerated to a point $\boldsymbol{Q}$, the second sub-equation in Eq. (5.1) is not valid; that is, we have an under-determined 2×3 linear system.

$$\begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{x} - \boldsymbol{Q} \end{pmatrix} \boldsymbol{\mathcal{B}} = \begin{pmatrix} \boldsymbol{x} \cdot \boldsymbol{x}' \\ \frac{|\boldsymbol{x}|^2 - \boldsymbol{Q}^2}{2} \end{pmatrix}, \quad (5.9)$$

Geometrically, there are now only two planes, intersecting into lines on the bisector, and thus the bisector is actually a ruled surface with the intersecting lines as its generators. There are two different approaches to solve this linear system (Figs. 4 and 5 compare the two approaches).

The Explicit Directrix Approach In [10], a third axillary plane that passes through the space point and has the corresponding generator as its normal is added to the linear system so that the directrix of the ruled bisector surface can be solved explicitly. Since [10] does not discuss the degrees, especially for the rational case, we will go into a little detail here.

The directrix $\mathfrak{D}$ is the solution to the following linear system,

$$\begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{x}' \times (\boldsymbol{x} - \boldsymbol{Q}) \\ \boldsymbol{x} - \boldsymbol{Q} \end{pmatrix} \mathfrak{D} = \begin{pmatrix} \boldsymbol{x} \cdot \boldsymbol{x}' \\ \boldsymbol{Q} \cdot (\boldsymbol{x}' \times (\boldsymbol{x} - \boldsymbol{Q})) \\ \frac{|\boldsymbol{x}|^2 - \boldsymbol{Q}^2}{2} \end{pmatrix}. \quad (5.10)$$

Assuming $\boldsymbol{x}(s)$ is a degree d *rational* B-spline, focusing only on degrees in s, we have,

$$\begin{pmatrix} 2d & 2d & 2d \\ 3d & 3d & 3d \\ d & d & d \end{pmatrix} \mathfrak{D} = \begin{pmatrix} 3d \\ 3d \\ 2d \end{pmatrix}. \quad (5.11)$$

The directrix $\mathfrak{D}$ is therefore symbolically computed to be a degree $(3d+3d+2d)+(2d+3d+d) = 14d$ rational, by the same procedure as we derived the degree of $\boldsymbol{\mathcal{B}}$ (cf.Eq. (5.1)) in Section 5.1. Because the generator is $\mathfrak{G} = (\boldsymbol{x}' \times (\boldsymbol{x} - \boldsymbol{Q}))$, which is a degree $3d$ rational[11], the final ruled bisector, symbolically computed from

$$\mathfrak{B} = \mathfrak{D} + t\mathfrak{G},$$

is a rational of degree $17d$.

Using the same strategy as we did in Section 5.1, Eq. (5.10) has a polynomial re-formulation,

$$\begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_1 \times (\boldsymbol{p} - w\boldsymbol{Q}) \\ \boldsymbol{p} - w\boldsymbol{Q} \end{pmatrix} w\mathfrak{D} = \begin{pmatrix} \boldsymbol{p} \cdot \mathbf{D}_1 \\ w\boldsymbol{Q} \cdot (\mathbf{D}_1 \times (\boldsymbol{p} - w\boldsymbol{Q})) \\ \frac{\boldsymbol{p}^2 - w\boldsymbol{Q}^2}{2} \end{pmatrix}, \quad (5.12)$$

and,focusing only on degree in s, this is,

$$\begin{pmatrix} 2d-1 & 2d-1 & 2d-1 \\ 3d-1 & 3d-1 & 3d-1 \\ d & d & d \end{pmatrix} w\mathfrak{D} = \begin{pmatrix} 3d-1 \\ 4d-1 \\ 2d \end{pmatrix}.$$

Consequently, $w\mathfrak{D}$ is the division of a degree $(2d-1)+(3d-1)+d + d = 7d-2$ polynomial B-spline by another degree $(2d-1)+(3d-1)+d = 6d-2$ polynomial B-spline. Noticing that $\mathfrak{D} = \frac{w\mathfrak{D}}{w}$, $\mathfrak{D}$ is the division of two polynomial B-spline, both having degree $7d-2$; therefore, $\mathfrak{D}$ is a degree $7d-2$ rational B-spline. Rewriting the generator as $\mathfrak{G} = \mathbf{D}_1 \times (\boldsymbol{x} - \boldsymbol{Q})$ with degree $3d-1$ in s, the final degree of the bisector is $10d-3$ in s.

Direct Approach The directrix of the ruled point/curve bisector, however, does not have to be solved explicitly; and also, for a fixed ruled surface, there are infinitely many choices for the directrix. In fact, Eq. (5.9) of course can be turned into a 2×2 linear system of two variable by moving any of the 3 components of $\boldsymbol{\mathcal{B}}$ to the RHS. Let us assume, for the moment, that the last component of $\boldsymbol{\mathcal{B}}$ can be moved to the RHS without introducing additional singularity to the linear system. Eq. (5.9) is transformed into,

$$\begin{pmatrix} \tilde{\boldsymbol{x}}' \\ \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{Q}} \end{pmatrix} \tilde{\boldsymbol{\mathcal{B}}} = \begin{pmatrix} \boldsymbol{x} \cdot \boldsymbol{x}' - \boldsymbol{x}'[2]\boldsymbol{\mathcal{B}}[2] \\ \frac{|\boldsymbol{x}|^2 - \boldsymbol{Q}^2}{2} - (\boldsymbol{x}[2] - \boldsymbol{Q}[2])\boldsymbol{\mathcal{B}}[2]. \end{pmatrix},$$

where the notation $\tilde{\boldsymbol{a}}$, for any vector $\boldsymbol{a}$, denotes the vector $\boldsymbol{a}$ with the last component discarded. Focusing only on degrees in s, this is,

$$\begin{pmatrix} d-1 & d-1 \\ d & d \end{pmatrix} \tilde{\boldsymbol{\mathcal{B}}} = \begin{pmatrix} 2d-1 \\ 2d \end{pmatrix}. \quad (5.13)$$

Notice that the above equation is actually the same as Eq. (5.8), considering the fact that the second row of the LHS matrix is all 0's there. Consequently, the bisector is

[11] or, a degree $3d$ polynomial if only the denominator vector is taken as the generator. This, however, does not change the final degree of the bisector

a rational of degree $3d - 1$ in s if the curve is a degree d polynomial. On the other hand, if the curve is a degree d rational, we would have a polynomial re-formulation (omitted here) just as we did in Section 5.1, and once again, the resulting bisector has a degree $4d - 1$ in s.

At this point, a few words are in order, comparing our direct approach method to the explicit directrix method in [10].

1. the degree of the final bisector is considerably reduced; compare Fig. 4(a) to Fig. 4(b) and Fig. 5(a) to Fig. 5(b).

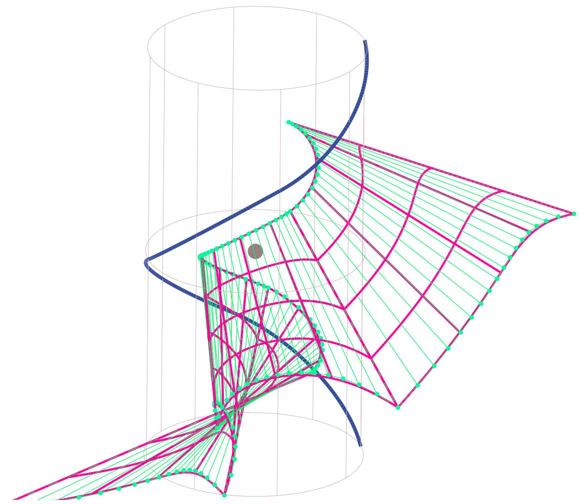
2. the iso curves of the ruled surface is exactly the height contour line; see Fig. 4(b) and 5(b)

3. more than one parameterization may have to be used to cover the whole bisector surface if moving some component (the last one in the derivation) does not work for the whole parametric space (In this case, the curve has to be splitted into separate segments. For any neighboring segment pair, different components of the bisector are moved to the RHS of Eq. (5.9)).

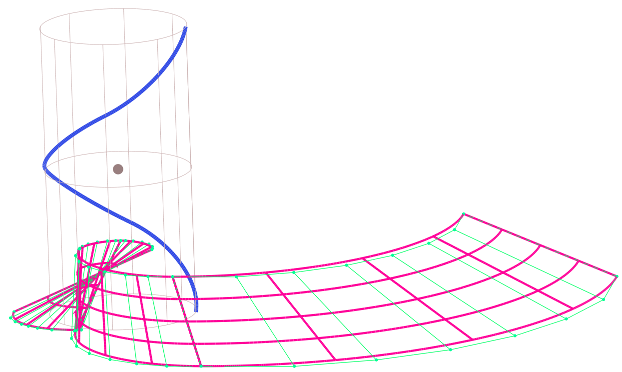
4. both approaches have the special case of plane bisector curve as an iso-curve of the ruled bisector surface; see Fig. 6.

6 Conclusion

In this paper, we have presented several degree reduction strategies for NURBS symbolic computation on *curves*, including eliminating higher order derivative terms, canceling common scalar factors, and developing symbolic computation as much as possible using *polynomial* B-splines. Although only focusing on NURBS symbolic computation, we would like to point out that there is also a numeric issue involved here, i.e. the effect of discarding the denominator to the final zero finding of the original rational. Fig. 2 gives us an example that the transformed and degree reduced B-spline even has a better numeric condition. We suspect, however, that the opposite might be true if both the numerator and the (discarded) denominator evaluated to close to zero at some points. In the future, we will extend the work in this paper to the surface case, especially to investigate any possible degree reducing reformulation for various surface interrogation problems.



(a) The Explicit Directrix Approach (final degree = 17)



(b) The Direct Approach (final degree = 7)

Figure 4. Point/Helix-Like-Curve Bisector

Comparing the two approaches to computing the ruled bisector between a space point (in gray) and a helix-like quadratic space curve. Control polygons in both images are in green.

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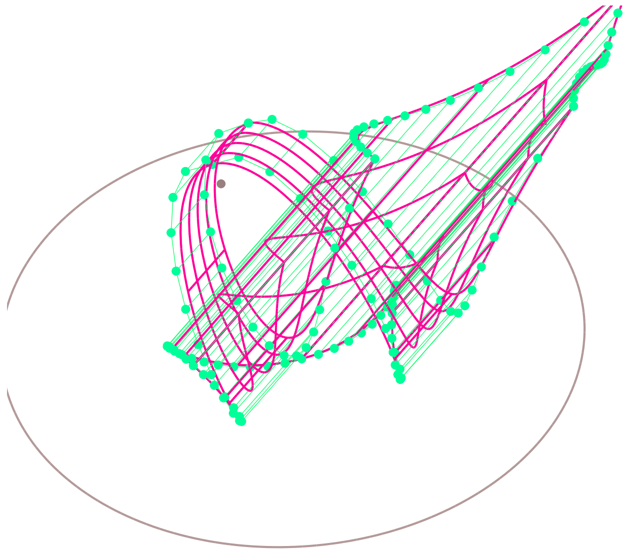
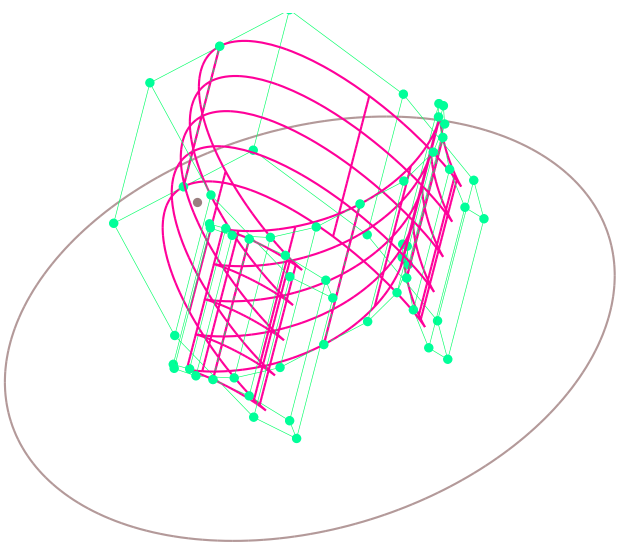


(a) The Explicit Directrix Approach (final degree = 17)



(b) The Direct Approach (final degree = 7)

Figure 5. Point/Ellipse Ruled Bisector

Comparing the two approaches to computing the ruled bisector between a plane point (in gray) and an ellipse. Control polygons in both images are in green.

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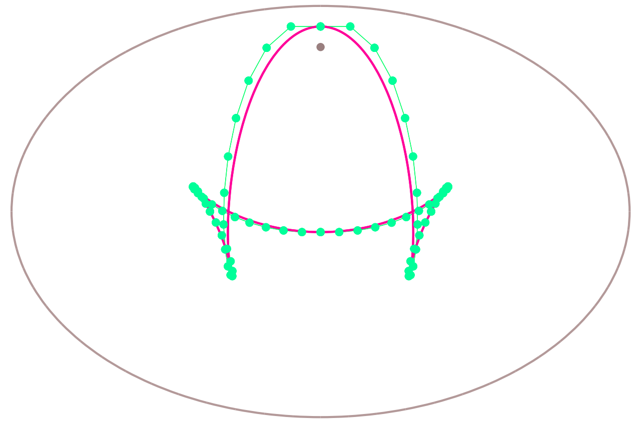
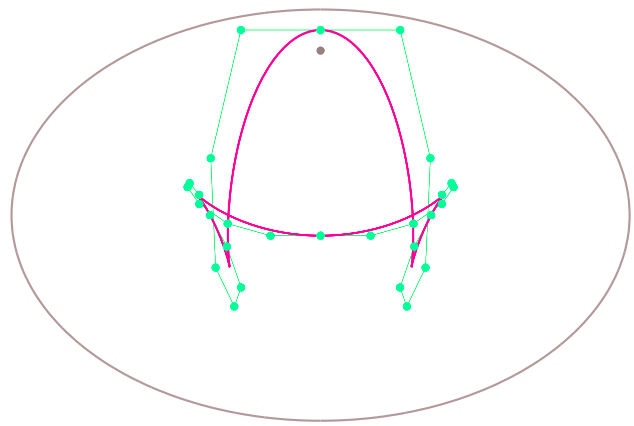


(a) The Explicit Directrix Approach (cf. Fig. 5(a))



(b) The Direct Approach (cf. Fig. 5(b))

Figure 6. The Plane Bisector Curve

The plane bisector curve in Fig 5 is the iso-curve of the ruled bisector for both approaches.

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