Rational Bézier Patch Differentiation using the Rational Forward Difference Operator[*]

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Abstract

This paper introduces the rational forward difference operator for differential computation on a rational Bézier patch based on its control mesh. With this rational version of the forward difference operator, and by ignoring the appropriate dependent (on lower order derivatives) terms of various derivatives, the derivatives themselves have the same expressions as their polynomial counterparts; the curvature and torsion, and the first and second fundamental forms, all have very similar expressions as those equations from classical differential geometry. This new approach also allows straightforward generalization to higher dimensional rational Bézier patches, the special co-dimension 1 case of which is treated with the result of the first and second fundamental forms.

Keywords*: forward difference, rational forward difference, rational Bézier patches, curvature, torsion, higher dimension rational Bézier patches.*

1 Introduction

Let $x(t)$ be a polynomial Bézier space curve in R^3 of degree n, and $b[t_1, \cdots, t_n]$ its blossom. From the theory of blossoming [7], we have the following equation for its r-th order derivative,

$$x_r = \frac{d^r x(t)}{dt^r} = \frac{n!}{(n-r)!} b[\vec{1}^r,\ t^{n-r}],\ \ r = 0, \cdots, n, \tag{1.1}$$

where the superscripts r and $n-r$ in the blossom mean that the corresponding arguments are repeated r and $n-r$ times, respectively.

Since $\vec{1} = 1 - 0$, evaluations with respect to vector $\vec{1}$ are manifested as differences. Using this, the above equation simply becomes,

$$x_r = \frac{n!}{(n-r)!} \Delta_r,\ \ r = 0, \cdots, n. \tag{1.2}$$

when the LHS is evaluated at $t = 0$, which corresponds to evaluating the RHS at the 0-th control point. The Δs are defined in the usual recursive manner,

$$\begin{aligned} &\Delta_0[i] = P_i,\ \ i = 0, 1, \cdots, n. \\ &\Delta_r[i] = \Delta_{r-1}[i+1] - \Delta_{r-1}[i], i = 0, \cdots, n-r. \end{aligned} \tag{1.3}$$

Below is the analogous equation for a polynomial Bézier surface, $x(u, v)$, of degree m in u and n in v. The mixed partial derivative of order $r + s$ is,

$$\begin{aligned} x_{rs} &= \frac{\partial^{r+s} x(u,v)}{\partial u^r \partial v^s} \\ &= \frac{m!}{(m-r)!} \frac{n!}{(n-s)!} b[\vec{1}^r\ u^{m-r} | \vec{1}^s\ v^{n-s}]. \end{aligned}$$

By analogously defining the forward difference for 2 dimensional case,

$$\begin{aligned} &\Delta_{00}[i][j] = P_{ij},\ \ i = 0, \cdots, m,\ \ j = 0, \cdots, n. \\ &\Delta_{rs}[i][j] = \Delta_{r-1,s}[i+1][j] - \Delta_{r-1,s}[i][j],\ \text{or}, \\ &\Delta_{rs}[i][j] = \Delta_{r,s-1}[i][j+1] - \Delta_{r,s-1}[i][j], \\ &\qquad i = 0, \cdots, m-r,\ \ j = 0, \cdots, n-s. \end{aligned} \tag{1.4}$$

we have,

$$x_{rs} = \frac{m!}{(m-r)!} \frac{n!}{(n-s)!} \Delta_{rs}, \tag{1.5}$$

where all expressions are evaluated at $(u, v) = (0, 0)$ or at the $(0, 0)$-th control point.

Equations (1.2) and (1.5) can be easily generalized to arbitrary dimension m, with degree n_i in the i direction.

$$x_{r_0, \cdots, r_{m-1}} = \prod_{i=0}^{0} \frac{n_i!}{(n_i - r_i)!} \Delta_{r_0, \cdots, r_{m-1}}, \tag{1.6}$$

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where $\Delta_{r_0,\cdots,r_{m-1}}$, the m dimensional forward difference, is defined analogously as Eq. (1.3) and Eq. (1.4).

In this paper, we use subscripts as a multi-index to mean the (partial) derivative with respect to the corresponding parameter. For example, in the above equation, x_{rs} means a mixed partial derivative of order r with respect to parameter t_0 (or u) and of order s with respect to parameter t_1 (or v). However, we maintain the common usage that a parameter subscript just means order 1 derivative with respect to that parameter. So, x_{uu} or $x_{t_0t_0}$ is the same as x_{20}. If the total derivative has a high order, the multi-index notation is more succinct, while the parameter notation is more common for low order derivatives. This remark applies to forward differences as well.

We have very simple derivative equations (Eq. (1.2) (1.5) and (1.6)) for polynomial Bézier patches. For the rational case, however, things become more involved and the derivative equations more complicated because the quotient rule for derivatives generates a denominator of increasing degree. In what follows, we offer an alternative approach that avoids this problem.

First we make the simple but critical observation that any dependent (i.e., spanned by all derivatives of lower orders) term of a derivative actually has *no* effect on the torsion (for co-dimension 2 manifolds) and various curvatures (including curve curvature, normal curvature, Gaussian curvature, mean curvature, etc.) calculation. Therefore we can omit it (or for that matter, add in some additional arbitrary dependent term, although we have not yet had to do that in this paper). For example, any dependent (on x') term of x'' in the curvature equation $\frac{\|x'\times x''\|}{\|x'\|^3}$ makes no contribution, and nor does any term of x''' in the torsion equation $\frac{[x'x''x''']}{\|x'\times x''\|^2}$ that is dependent on $\{x', x''\}$ (See Section 3). Here, $[x'x''x'''] = (x'\times x'')\cdot x'''$, is the triple scalar product of x', x'' and x'''. Similar observations apply to any terms of x_{uv} which are spanned by x_u and x_v (i.e., in the tangent plane) in the computation of $M = \hat{N}\cdot x_{uv}$, etc.(see Section 4).

Throughout this paper, we use $[E_1] = [E_2]$ notation if derivative vectors E_1 and E_2 yield equivalent results in calculating aforementioned curvature or torsion. Basically it is just the equivalent class of all vectors having the same orthogonal (to the subspace spanned by all the lower order derivatives) components, and we can choose any one of them as the representative, which usually is not the original derivative, but not necessarily the original derivative with all the dependent component eliminated either.

For our development, we will need to define a rational version of the (polynomial) forward difference operator.

Definition 1 (Rational Forward Differences) *Suppose a rational Bézier patch of dimension m, of degree n_i in the i-th direction, embedded in Euclidean space of dimension $q > m$, is defined by an m-dimensional control mesh with homogeneous control points of (P_s, W_s), where s is the multi-index $s_0s_1\cdots s_{m-1}$, $s_i = 0, 1, \cdots, n_i$, and $P_s \in R^q$, $W_s \in R$.*

The rational forward difference operator of order r_i in the i-th direction, is defined at the s-th control point, for $s_i = 0, 1, \cdots, n_i - r_i$ and $i = 0, 1, \cdots, m-1$, as

$$\bar{\Delta}_r(P, W)[s] = \frac{\Delta_r(P)[s]W_s - \Delta_r(W)[s]P_s}{W_s^2},$$

where r stands for $r_0r_1\cdots r_{m-1}$, $\Delta_r(P)[s]$ and $\Delta_r(W)[s]$ denote the usual (polynomial) forward difference operators evaluated at the s-th control point.

Combining this rational forward difference operator in Definition 1 and the observation preceding it, this paper derives consistent, virtually equivalent rational forms for the derivatives as those known in the Bézier polynomial case and also similar forms for curvature and torsion, the first and second fundamental forms, as those from classical differential geometry. This is not only gratifying in terms of mathematical elegance; but more importantly it provides a straightforward generalization to higher dimension and translates into very simple and unified code for differential geometry computation on NURBS manifolds of arbitrary dimension. This benefit distinguishes the approach described herein.

Without loss of generality, in this paper, all derivatives (forward differences) are evaluated at the lower corner of the domain (the control mesh), and therefore we develop differential computation on only one point of the patch. Employing subdivision techniques, however, the same differential features clearly can be extracted from any point on a NURBS manifold. Further, we note that curvature and torsion are intrinsic geometric properties inherited by each subdivided curve part, during a subdivision process, so it is not necessary to recompute this value subsequently at any time in the recursion. That is to say, if we subdivide a Bézier curve into two smaller pieces, the geometric properties at the left hand point are inherited by that of the first piece, so only the second piece requires evaluation at its left hand point. In other words, this leads directly to an incremental algorithm that arises as a by-product of the subdivision process typically used for the visualization of the manifold.

The rest of the paper is organized as follows. First we give a comparative discussion of related works in Section 2, derive differential computation on rational Bézier curves and surfaces in Section 3 and Section 4 respectively. Then we give two examples in Section 5. After that, in Section 6, we generalize to rational Bézier hyper-surfaces. Finally, looking backward and forward, we make some concluding remarks in Section 7.

2 Related Works

Curvature computation for rational Bézier curves based on control meshes is well known [5, 4, 1]. This, however, is not the case for rational Bézier surfaces. J. Zheng and T. Sederberg [8] published some earlier result on this with good geometric intuition. Inspired by their paper, our work is in a similar vein. However, we introduce the rational forward difference operation, and use that as a simplifying approach to achieve succinct expressions that are unifiable over manifold dimension. The rational forward difference approach allows us straightforwardly to generalize to higher dimensional situation, an extension which we touch on in Section 6. Moreover, from an implementation viewpoint, another advantage is that the simpler formulation translates directly to clean, compact code.

3 Rational Bézier Curves

In this section, we derive formulas for derivatives, curvature and torsion of a rational Bézier curve, all expressed in rational forward difference operators. Throughout this section, we assume the degree n rational Bézier curve is defined by a control polygon with homogeneous points $(P_0, W_0), (P_1, W_1), \cdots, (P_n, W_n)$.

Notice in all the work that follows, the degree of the denominator of any higher order derivative stays the same as that of the first order derivative; a higher degree denominator comes only with terms that are in the subspace spanned by all the lower order derivative and can therefore be eliminated. This remark applies to the surface case in Section 4 and the hyper-surface case in Section 6 as well.

3.1 First Order Derivatives

Apply the quotient rule to a given rational parametrization,

$$
\begin{aligned}
x(t) &= \frac{p(t)}{w(t)},\\
x'(t) &= \frac{(p'w - w'p)}{w^2},
\end{aligned}
\tag{3.1}
$$

at $t = 0$, i.e., the starting point of the Bézier curve, we have (cf. Eq (1.2)),

$$
\begin{aligned}
p' &= n(P_1 - P_0) = n\Delta_1(P),\\
w' &= n(W_1 - W_0) = n\Delta_1(W),\\
x' &= n\frac{\Delta_1(P)W_0 - \Delta_1(W)P_0}{W_0^2}
\end{aligned}
$$

Then, by Definition 1,

$$
x' = n\bar{\Delta}_1. \tag{3.2}
$$

3.2 Second Order Derivatives

Differentiating Eq. (3.1) one more time,

$$
[x''] = \frac{(p''w - w''p)}{w^2}, \tag{3.3}
$$

where the term due to the derivative of the denominator (and therefore is parallel to x') has already been discarded.

Again with Definition 1,

$$
[x''] = n(n-1)\bar{\Delta}_2. \tag{3.4}
$$

Note that we should also have put a pair of brackets [ ] around the right side expression, meaning that x'' and $n(n-1)\bar{\Delta}_2$ are in the same equivalent class defined by having the same orthogonal component. For the sake of simpler math expressions, however, we accept this inconsistency for this equation and all similar equations in the rest of the paper.

3.3 Third Order Derivatives

It turns out that the equation for the third order derivative, $[x''']$, also provides a simple formulation.

Differentiating Eq. (3.3) one more time, we get,

$$
[x'''] = \frac{p'''w - w'''p}{w^2} + \frac{p''w' - w''p'}{w^2}, \tag{3.5}
$$

where, again, the term due to the derivative of the denominator(parallel to x'') has already been discarded.

If we can prove that the second term of the above equation is spanned by (x', x''), or $(p'w - w'p,\ p''w - w''p)$ (cf. Eq. (3.1) and Eq. (3.3)), then we would have

$$
[x'''] = n(n-1)(n-2)\bar{\Delta}_3, \tag{3.6}
$$

where, $\bar{\Delta}_3$ is the third order rational forward difference.

Now, to prove that $p''w' - w''p'$ is spanned by $p'w - w'p$ and $p''w - w''p$, we take the triple scalar product of these 3 vectors, and show it vanishes. Specifically [1]

$$(p'w - w'p) \times (p''w - w''p) \cdot (p''w' - w''p') =$$
$$((p' \times p'')w^2 - (p \times p'')ww' - (p' \times p)ww'') \cdot (p''w' - w''p')$$
$$= (p \times p'')ww' \cdot w''p' - (p' \times p)ww'' \cdot p''w'$$
$$= 0$$

3.4 The Curvature and Torsion of Rational Bézier Curves

For rational Bézier curves, rational forward difference operators really allow a direct translation of curvature and torsion equations from classical differential geometry [6, 2].

$$\begin{aligned}\kappa &= \frac{\|x' \times x''\|}{\|x'\|^3} = \frac{\|x' \times [x'']\|}{\|x'\|^3} \\ &= \frac{n-1}{n}\frac{\|\bar{\Delta}_1 \times \bar{\Delta}_2\|}{\|\bar{\Delta}_1\|^3}\end{aligned} \tag{3.7}$$

$$\begin{aligned}\tau &= \frac{[x'x''x''']}{\|x' \times x''\|^2} = \frac{[x'[x''][x''']]}{\|x' \times [x'']\|^2} \\ &= \frac{n-2}{n}\frac{[\bar{\Delta}_1\bar{\Delta}_2\bar{\Delta}_3]}{\|\bar{\Delta}_1 \times \bar{\Delta}_2\|^2}\end{aligned} \tag{3.8}$$

Comparing to the common formulation of the curvature and torsion of *polynomial* Bézier curves [5, 4], the above equations have no explicit reference to the weights of the control points, and assume exactly the same forms, replacing the forward differences with the *rational* forward differences.

4 Rational Bézier Tensor Surfaces

In this section we work on differential computation on a rational Bézier tensor surface. Throughout this section, we assume the degree m (in u-direction) by n (in v-direction) rational Bézier tensor surface is defined by a control mesh with homogeneous points (P_{ij}, W_{ij}), where $i = 0, \cdots, m$ and $j = 0, \cdots, n$.

4.1 First Order Derivatives

Let the rational parametrization be,

$$x(u, v) = \frac{p(u, v)}{w(u, v)}.$$

[1] In these 3 expressions, w, w' and w'' are scalar values, p' and p'' are vectors; and $w'p$ and $w''p$ are also vectors, because they are actually difference of two points. Therefore, the subsequent derivation makes sense mathematically.

Then, the first order partial derivatives are,

$$\begin{aligned}x_u &= \frac{(p_u w - w_u p)}{w^2}, \\ x_v &= \frac{(p_v w - w_v p)}{w^2}.\end{aligned} \tag{4.1}$$

At $(u, v) = (0, 0)$, i.e., the lower left corner point of the rational Bézier surface, we have (cf. Eq (1.5)),

$$\begin{aligned}p_u &= m(P_{10} - P_{00}) = m\Delta_{10}(P), \\ p_v &= n(P_{01} - P_{00}) = n\Delta_{01}(P), \\ w_u &= m(W_{10} - W_{00}) = m\Delta_{10}(W), \\ w_v &= n(W_{01} - W_{00}) = n\Delta_{01}(W),\end{aligned}$$

and consequently,

$$\begin{aligned}x_u &= m\frac{\Delta_{10}(P)W_{00} - \Delta_{10}(W)P_{00}}{W_{00}^2} = m\bar{\Delta}_{10}, \\ x_v &= n\frac{\Delta_{01}(P)W_{00} - \Delta_{01}(W)P_{00}}{W_{00}^2} = n\bar{\Delta}_{01}.\end{aligned} \tag{4.2}$$

4.2 Second Order Derivatives

For the non-mixed second order derivatives, things are almost the same as the curve case,

$$\begin{aligned}[x_{uu}] &= m(m-1)\bar{\Delta}_{20}, \\ [x_{vv}] &= n(n-1)\bar{\Delta}_{02}.\end{aligned} \tag{4.3}$$

Now, to deal with the mixed partial derivative by taking one more partial derivative of Eq. (4.1) with respect to v, we have,

$$[x_{uv}] = \frac{(p_{uv}w - w_{uv}p)}{w^2} + \frac{(p_u w_v - w_u p_v)}{w^2}.$$

Verifying that the triple scalar product of $p_u w_v - w_u p_v$ with $p_u w - w_u p$ and $p_v w - w_v p$ vanishes,

$$((p_u w - w_u p) \times (p_v w - w_v p)) \cdot (p_u w_v - w_u p_v) =$$
$$(w^2 p_u \times p_v - w_u wp \times p_v - ww_v p_u \times p) \cdot (p_u w_v - w_u p_v)$$
$$= -w_u ww_v[pp_v p_u] + ww_v w_u[p_u pp_v]$$
$$= 0,$$

we can eliminate the second term in the above equation, and again get the appealingly expression,

$$[x_{uv}] = mn\bar{\Delta}_{11}. \tag{4.4}$$

4.3 The First and Second Fundamental Forms of Bézier Surfaces

Interpreting the classical first fundamental form for a surface, it becomes,

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{pmatrix}$$

$$= \begin{pmatrix} mm\bar{\Delta}_{10} \cdot \bar{\Delta}_{10} & mn\bar{\Delta}_{10} \cdot \bar{\Delta}_{01} \\ nm\bar{\Delta}_{01} \cdot \bar{\Delta}_{10} & nn\bar{\Delta}_{01} \cdot \bar{\Delta}_{01} \end{pmatrix}. \tag{4.5}$$

Also denoted as I, the determinant is

$$\begin{aligned} I &= m^2n^2\|\bar{\Delta}_{10} \times \bar{\Delta}_{01}\|^2 \\ &= m^2n^2A^2, \end{aligned} \tag{4.6}$$

where, $A = \|\bar{\Delta}_{01} \times \bar{\Delta}_{10}\|$.

In preparation for the second fundamental form, we normalize the surface normal $N = x_u \times x_v$ to,

$$\hat{N} = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{\bar{\Delta}_{10} \times \bar{\Delta}_{01}}{\|\bar{\Delta}_{10} \times \bar{\Delta}_{01}\|}.$$

And the second fundamental form becomes,

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} x_{uu} \cdot \hat{N} & x_{uv} \cdot \hat{N} \\ x_{vu} \cdot \hat{N} & x_{vv} \cdot \hat{N} \end{pmatrix}$$

$$= \begin{pmatrix} m(m-1)\frac{[\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{20}]}{\|\bar{\Delta}_{10}\times\bar{\Delta}_{01}\|} & mn\frac{[\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{11}]}{\|\bar{\Delta}_{10}\times\bar{\Delta}_{01}\|} \\ nm\frac{[\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{11}]}{\|\bar{\Delta}_{10}\times\bar{\Delta}_{01}\|} & n(n-1)\frac{[\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{02}]}{\|\bar{\Delta}_{10}\times\bar{\Delta}_{01}\|} \end{pmatrix}$$

$$= \begin{pmatrix} m(m-1)\frac{V_{20}}{A} & mn\frac{V_{11}}{A} \\ mn\frac{V_{11}}{A} & n(n-1)\frac{V_{02}}{A} \end{pmatrix}, \tag{4.7}$$

where,

$$\begin{aligned} V_{02} &= [\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{02}], \\ V_{11} &= [\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{11}], \\ V_{20} &= [\bar{\Delta}_{10}\bar{\Delta}_{01}\bar{\Delta}_{20}]. \end{aligned}$$

The determinant of the second fundamental form matrix, also denoted as II, is,

$$\begin{aligned} II &= LN - M \\ &= \frac{mn(m-1)(n-1)V_{20}V_{02} - m^2n^2V_{11}^2}{A^2}. \end{aligned} \tag{4.8}$$

And finally, the Gaussian curvature is,

$$K = \frac{II}{I} = \frac{\frac{(m-1)(n-1)}{mn}V_{20}V_{02} - V_{11}^2}{A^4}. \tag{4.9}$$

We omit the derivations of other second order surface features, such as mean curvature, principal curvatures, and asymptotic directions, since they are easily derived from the first and second fundamental forms.

5 Examples

To illustrate the main idea of this paper, we choose the simple curve case to give two examples in this section.

The first example computes the curvature of a rational conic section. The second example does the same work on the same curve, but within the context of NURBS.

Example 1 (Curvatures of an ellipse via the rational forward operator *Consider the first quadrant of the ellipse*

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1,$$

with parametrization of

$$\begin{aligned} x_1(t) &= \frac{a \times (1-t^2)}{1+t^2}, \\ x_2(t) &= \frac{b \times 2t}{1+t^2}, \end{aligned}$$

at the interval $[0, 1]$.

Rewriting the parametrization in homogenous coordinate form, we have,

$$x(t) = [a(1-t^2),\ \ 2bt,\ \ 1+t^2].$$

Using blossoming, we can easily find the rational Bézier control points of this conic section in homogenous coordinates,

$$\begin{aligned} [P0, W0] &= [a, 0, 1], \\ [P1, W1] &= [a, b, 1], \\ [P2, W2] &= [0, 2b, 2]. \end{aligned}$$

The first order forward differences at the first control point are,

$$\begin{aligned} \Delta_1(P)[0] &= P1 - P0 = [a, b] - [a, 0] = [0, b], \\ \Delta_1(W)[0] &= W1 - W0 = 0. \end{aligned}$$

By Definition 1, the first order rational forward difference at the first control point is,

$$\bar{\Delta}_1[0] = \frac{\Delta_1(P)[0]W0 - \Delta_1(W)[0]P0}{W0^2} = \frac{[0, b] + 0[a, 0]}{1} = [0, b]$$

The first order forward differences at the second(middle) control point are,

$$\Delta_1(P)[1] = P2 - P1 = [0, 2b] - [a, b] = [-a, b],$$
$$\Delta_1(W)[1] = W2 - W1 = 1.$$

Therefore, the second order forward differences at the first control point are,

$$\Delta_2(P)[0] = \Delta_1(P)[1] - \Delta_1(P)[0] = [-a, b] - [0, b] = [-a, 0],$$
$$\Delta_2(W)[0] = \Delta_1(W)[1] - \Delta_1(W)[0] = 1,$$

Now again by Definition 1, the second order rational forward difference at the first control point is,

$$\bar{\Delta}_2[0] = \frac{\Delta_2(P)[0]W0 - \Delta_2(W)[0]P0}{W0^2} = \frac{[-a, 0] - [a, 0]}{1} = [-2a, 0]$$

Based on the first and second order rational differences,

$$[x'(0)] = 2 \times \bar{\Delta}_1[0] = [0, 2b],$$
$$[x''(0)] = 2 \times (2-1)\bar{\Delta}_2[0] = [-4a, 0].$$

And, by Eq. (3.7)*, the curvature of the ellipse at* $t = 0$*, i.e., the major end, is,*

$$\kappa = \frac{n-1}{n}\frac{\|\bar{\Delta}_1 \times \bar{\Delta}_2\|}{\|\bar{\Delta}_1\|^3} = \frac{2ab}{2b^3} = a/b^2$$

Although we define the rational forward difference operator and its application in the context of Bézier patches, the main idea actually extends to NURBS as shown in the following example.

Example 2 (Rational forward differences in the context of NURBS) *Consider the same ellipse as the above example, blossoming, however, the explicit parametrization using the knot vector of* $-1, 0, 1, 2$ *(instead of* $0, 0, 1, 1$*, which essentially results in a Bézier curve) so that the de Boor control points are,*

$$[P0, W0] = [a, -b, 1],$$
$$[P1, W1] = [a, b, 1],$$
$$[P2, W2] = [-a, 3b, 3].$$

The first order forward differences at the first control points are,

$$\Delta_1(P)[0] = \frac{P1 - P0}{1-(-1)} = \frac{[a, b] - [a, -b]}{2} = [0, b],$$
$$\Delta_1(W)[0] = \frac{W1 - W0}{1-1} = 0.$$

Note that the polynomial forward difference has to be divided by $1-(-1)$ *, which normalizes* $\vec{1} - (\vec{-1}) = \vec{2}$ *back to* $\vec{1}$ *(cf. Eq.* (1.1)*). Nonetheless, we can use Definition 1 as usual to get the first order rational forward difference at the first control point,*

$$\bar{\Delta}_1[0] = \frac{\Delta_1(P)[0]W0 - \Delta_1(W)[0]P0}{W0^2} = \frac{[0, b] - 0[a, -b]}{1^2} = [0, b]$$

The first order forward differences at the second(middle) control point are,

$$\Delta_1(P)[1] = \frac{P2 - P1}{2-0} = \frac{[-a, 3b] - [a, b]}{2} = [-a, b],$$
$$\Delta_1(W)[1] = \frac{W2 - W1}{2-0} = 1.$$

Therefore, the second order forward differences at the first control point are,

$$\Delta_2(P)[0] = \frac{\Delta_1(P)[1] - \Delta_1(P)[0]}{1-0} = [-a, b] - [0, b] = [-a, 0],$$
$$\Delta_2(W)[0] = \frac{\Delta_1(W)[1] - \Delta_1(W)[0]}{1-0} = 1.$$

Now again by Definition 1, the second order rational forward difference at the first control point is,

$$\bar{\Delta}_2[0] = \frac{\Delta_2(P)[0]W0 - \Delta_2(W)[0]P0}{W0^2} = \frac{[-a, 0] - [a, -b]}{1^2} = [-2a, b]$$

Based on the first and second order rational difference,

$$[x'(0)] = 2 \times \bar{\Delta}_1[0] = [0, 2b],$$
$$[x''(0)] = 2 \times (2-1) \times [-2a, b] = [-4a, -2b].$$

And, by Eq. (3.7)*, the curvature of the ellipse at* $t = 0$*, i.e., the major end, is,*

$$\kappa = \frac{n-1}{n}\frac{\|\bar{\Delta}_1 \times \bar{\Delta}_2\|}{\|\bar{\Delta}_1\|^3} = \frac{2ab}{2b^3} = a/b^2$$

Note that, compared to Example 1, we have the same result for $[x'(0)]$ *(*$\bar{\Delta}_1[0]$*), but different one for* $[x''(0)]$ *(*$\bar{\Delta}_2[0]$*). However, there is actually no inconsistency here, as we are talking about the the second derivative up to any term parallel to the first derivative. Both* $[x''(0)]$ *(*$\bar{\Delta}_2[0]$*) can be used to compute the curvature.*

6 Rational Bézier Hyper-Surfaces

Differential computation on a rational Bézier manifold, with arbitrary dimension and co-dimension, is a formidable problem. With the rational forward difference operation introduced in this paper, we try to make a modest step toward this goal - the co-dimension 1 case.

Considering an m dimensional rational Bézier patch embedding in R^{m+1} space, the first fundamental form is an $m \times m$ matrix, the (i, j)-th element of which is merely the dot product of the two partial derivatives along i and j directions,

$$I(ij) = x_{t_i} \cdot x_{t_j} = n_i n_j \bar{\Delta}_{t_i} \cdot \bar{\Delta}_{t_j}. \tag{6.1}$$

By analogous reasoning, the m partial derivatives $\{x_{t_i}, i = 0, \cdots, m-1\}$ span the tangent hyper-plane, and their cross product, defined as (in the equation, $\{e_0, \cdots, e_m\}$ is the orthonormal base of the embedding space, and $(x_{t_i})_j$ is the j-th component of x_{t_i}),

$$x_{t_0} \times \cdots \times x_{t_{m-1}} = \begin{vmatrix} e_0 & \cdots & e_m \\ (x_{t_0})_0 & \cdots & (x_{t_0})_m \\ \vdots & \ddots & \vdots \\ (x_{t_m})_0 & \cdots & (x_{t_m})_m \end{vmatrix},$$

gives the normal direction of the hyper-surface. For the non-mixed second order derivatives, it is obvious we have the same result as the surface case or as the (iso-parametric) curve case. For the mixed second order partial derivatives, although the notations are more complicated, we are able to make a similar argument below as we did in the surface case for x_{uv}.

$$\begin{aligned} x_{t_i} &= \frac{\partial}{\partial t_i}\left(\frac{p(t_0, \cdots, t_{m-1})}{w(t_0, \cdots, t_{m-1})}\right) \\ &= \frac{p_{t_i} w - w_{t_i} p}{w^2}, \end{aligned}$$

$$[x_{t_i t_j}] = \frac{(p_{t_i t_j} w - w_{t_i t_j} p)}{w^2} + \frac{(p_{t_i} w_{t_j} - w_{t_i} p_{t_j})}{w^2}.$$

Once again, the second term on the above equation is actually in the tangent hyper-plane and can be discarded. This is true because the $(m+1)$-th scalar product of the m partial derivatives $\{x_{t_i}, i = 0, \cdots, m-1\}$, and $v = (p_{t_i} w_{t_j} - w_{t_i} p_{t_j})$ defined as,

$$[x_{t_0} \cdots x_{t_{m-1}} \;\; v] = \begin{vmatrix} v_0 & \cdots & v_m \\ (x_{t_0})_0 & \cdots & (x_{t_0})_m \\ \vdots & \ddots & \vdots \\ (x_{t_m})_0 & \cdots & (x_{t_m})_m \end{vmatrix},$$

vanishes. Specifically, by observing that any multiple appearance of any vector in an $(m+1)$-th scalar product makes the product 0,

$$\begin{aligned} &[(p_{t_0} w - w_{t_0} p) \cdots (p_{t_{m-1}} w - w_{t_{m-1}} p) \;\; (p_{t_i} w_{t_j} - w_{t_i} p_{t_j})] \\ &= [(-w^{m-1} w_{t_i}) \;\; p_{t_0} \cdots p_{t_{i-1}} p \, p_{t_{i+1}} \cdots p_{t_{m-1}} \;\; (p_{t_i} w_{t_j})] - \\ &\quad [(-w^{m-1} w_{t_j}) \;\; p_{t_0} \cdots p_{t_{j-1}} p \, p_{t_{j+1}} \cdots p_{t_{m-1}} \;\; (p_{t_j} w_{t_i})] \\ &= 0. \end{aligned}$$

Hence, we have the familiar expressions for the second order partial derivatives,

$$\begin{aligned} &[x_{t_i t_i}] = n_i(n_i - 1)\bar{\Delta}_{t_i t_i}, i = 0, \cdots, m-1, \\ &[x_{t_i t_j}] = n_i n_j \bar{\Delta}_{t_i t_j}, i, j = 0, \cdots, m-1, \;\; i \neq j. \end{aligned} \tag{6.2}$$

and the second fundamental form,

$$\begin{aligned} II(ii) &= n_i(n_i - 1)\frac{V_{t_i t_i}}{A}, i = 0, \cdots, m-1, \\ II(ij) &= n_i n_j \frac{V_{t_i t_j}}{A}, i, j = 0, \cdots, m-1, \;\; i \neq j, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} V_{t_i t_j} &= [\bar{\Delta}_{t_0} \cdots \bar{\Delta}_{t_{m-1}} \bar{\Delta}_{t_i t_j}], \\ A &= \|\bar{\Delta}_{t_0} \times \bar{\Delta}_{t_1} \times \cdots \times \bar{\Delta}_{t_{m-1}}\|. \end{aligned}$$

7 Conclusion

In this paper, a rational version of the forward difference operator on control meshes is introduced. With the rational forward difference operation, it turns out that the basic differential geometry computation on rational Bézier patches results in very compact, simple formulas that specialize to previously known forms for their polynomial counterparts and resemble those from classical differential geometry. A generalization to co-dimension 1 hyper-surfaces is also discussed, and we believe that this formulation is more amenable if the Bézier manifold has arbitrary co-dimensions. This new approach also leads directly to simple, unified implementations that handle all dimensions within single algorithm.

Currently, we are considering the third and higher order differential computation on Bézier patches. The extension to higher (> 2) order derivatives is not straightforward, since the lower order derivatives already span the embedding Euclidean 3D space. For example, when deriving x_{uvv} by $x_{uvv} = \partial x_{uv}/\partial v$, the term due to the derivative of the denominator of x_{uv} can no longer be ignored in $x_{uvv} \cdot N$, which is used to compute flecnodal curves.

Another direction for extension is symbolic computation on NURBS (See G. Elber [3]). Specifically, we have successfully managed to eliminate the higher degree terms of the second and third order derivatives of a rational curve, as well as the second order derivatives of a rational surface, and we would like to explore its extension to some types of symbolic computation.

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