Theoretically Based Robust Algorithms for Tracking Intersection Curves of Two Deforming Parametric Surfaces ⋆

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Abstract. This paper applies singularity theory of mappings of surfaces to 3-space and the generic transitions occurring in their deformations to develop algorithms for continuously and robustly tracking the intersection curves of two deforming parametric spline surfaces, when the deformation is represented as a family of generalized offset surfaces. The set of intersection curves of 2 deforming surfaces over all time is formulated as an implicit 2-manifold $\mathcal{I}$ in the augmented (by time domain) parametric space $\mathbb{R}^5$. Hyper-planes corresponding to some fixed time instants may *touch* $\mathcal{I}$ at some isolated transition points, which delineate transition events, i.e., the topological changes to the intersection curves. These transition points are the 0-dimensional solution to a rational system of 5 constraints in 5 variables, and can be computed efficiently and *robustly* with a rational constraint solver using subdivision and hyper-tangent bounding cones. The actual transition events are computed by contouring the local osculating paraboloids. Away from any transition points, the intersection curves do not change topology and evolve according to a simple evolution vector field that is constructed in the *euclidean space* in which the surfaces are embedded.

1 Introduction and Related Work

In this paper, we consider the dynamic intersection of two deforming parametric surfaces. The surface deformation is represented by a family of generalized offset surfaces, which is an example of a "radial flow"of a generalized offset vector field introduced in [**?**, **?**] (also see [**?**] for a mathematically less technical discussion). This extends the standard unit normal offset surfaces. Specifically, let $\varsigma(s)$, $s \in \mathbb{R}^2$, be a parameterization of a regular initial surface; and let $U(s)$ denote an offset vector field (parameterized again by s). Such a U need be neither unit-length nor orthogonal to the tangent plane, but does not lie in the tangent plane. The generalized offset surface flow is defined by,

$$\sigma(s;t) = \varsigma(s) + tU(s); \tag{1}$$

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where $0 \leq t \leq 1$ is the offset time. Each of the two deforming surfaces is assumed to remain regular and be free of self-intersections throughout the deformation process. Conditions ensuring such regularity are given in [?] and [?].

Research into finding surface-surface intersections has mostly focused on the static problem [?, ?, ?, ?, ?], and the case of the unit normal offset surfaces [?, ?, ?, ?, ?, ?, ?, ?], We emphasize the topological robustness of surface intersection, which has been an important and extensively researched topic for static surface-surface intersection ([?, ?, ?, ?, ?, ?, ?]). In [?], Jun et al. worked on surface slicing, i.e., the intersection of a surface with a series of parallel planes, exploring the relation between the transition points and the topology of contour curves. The transition points, though, are used only to efficiently and robustly find the starting point of the contour curves for a marching algorithm [?, ?] to trace out the whole curve. Ouyang et al. [?] applied a similar approach to the intersection of two unit normal vector offset surfaces.

Applied to mappings of surfaces to $\mathbb{R}^3$, singularity theory [?] provides a theoretical classification of both the local stable properties of mappings of surfaces and of the generic transitions they undergo under deformation. Our assumptions on the regularity of the surfaces characterizes the transition of the intersection curves of the deforming surfaces to one of a list of standard generic transitions. Between transitions, the intersection curves evolve in a smooth way without undergoing topological transitions.

This paper is organized to deal with these two cases. In Section 2, we construct an evolution vector field which allows us to follow the evolution of intersection curves (ICs) by discretely solving a differential equation in the parametric space. In Section 3, we represent the locus of intersection curves of the two deforming surfaces as a 2–manifold $\mathcal{I}$ in a 5–dimensional *augmented parameter space*. In Section 4 we turn to the second problem of computing the transition events, and tracking the topological changes of the intersection curves occurring at transition points. In Section 4.1, we enumerate the generic transition points classified by singularity theory and provide an alternative characterization as critical points of a function on the implicit surface $\mathcal{I}$. This provides the theoretical basis to our algorithm that detects transition points as the simultaneous 0-set of a rational system of 5 constraints in 5 variables. Then, in Section 4.2 we compute the transitions in the intersection curves using contours on the local osculating quadric of the surface $\mathcal{I}$ at the critical points. A concluding discussion of the issues ensues in Section 5.

2 Evolution of Intersection Curves

Consider two deforming surfaces, σ and $\hat{\sigma}$, represented as generalized offset surfaces,

$$\sigma(s,t) = \varsigma(s) + t\ U(s),$$
$$\hat{\sigma}(\hat{s},t) = \hat{\varsigma}(\hat{s}) + t\ \hat{U}(\hat{s}),$$

where $s = (s_1, s_2) \in \mathbb{R}^2$ and $\hat{s} = (\hat{s}_1, \hat{s}_2) \in \mathbb{R}^2$ are the parameters of $\varsigma(s)$ and $\hat{\varsigma}(\hat{s})$, and their corresponding offset vector fields $U(s)$ and $(\hat{U}(\hat{s})$, respectively. We write the coordinate representation of the deforming surfaces by

$$\sigma(s,t) = (x(s),t), y(s,y), z(s,t)) \text{ and } \hat{\sigma}(\hat{s},t) = (\hat{x}(\hat{s},t), \hat{y}(\hat{s},t), \hat{z}(\hat{s},t)).$$

Define $\mathcal{L}_0$ to be the set of all points in $\mathbb{R}^3$ on a local intersection curves of the two deforming local surfaces over all times t. Consider a point P on an intersection curve of the two deforming surfaces at some time t. We first assume that the tangent planes to the two offset surfaces at P are different; otherwise, we are in the singular case corresponding to a transition event, which we will discuss in section 4. We use the notation $\hat{\sigma}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$ to denote the partial derivative $\frac{\partial \hat{\sigma}}{\partial \hat{s}_i} = (\frac{\partial \hat{x}}{\partial \hat{s}_i}, \frac{\partial \hat{y}}{\partial \hat{s}_i}, \frac{\partial \hat{z}}{\partial \hat{s}_i})$ $(i = 1, 2)$, and analogously for σ_i. Define

$$N = \sigma_1 \times \sigma_2, \quad \hat{N} = \hat{\sigma}_1 \times \hat{\sigma}_2$$

to be the 2 non-unit length normals to each of the two surfaces, respectively. Further let

$$\bar{N} = (N \times \hat{N}) \times \hat{N}.$$

to be the tangent vector of $\hat{\sigma}$ at P that is perpendicular to the intersection curve.

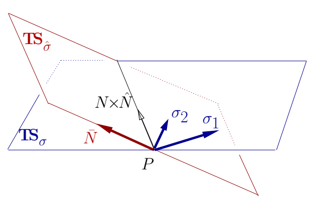


Fig. 1. Local Basis $\{\sigma_1, \sigma_2, \bar{N}\}$

Because the two tangent planes to the two surfaces at P are different, $\{\sigma_1, \sigma_2, \bar{N}\}$ is a basis of $\mathbb{R}^3$ (Fig. 1). Decomposing $\delta U = \hat{U} - U$ in this basis gives,

$$\delta U = \hat{U} - U = a\sigma_1 + b\sigma_2 + c\bar{N}$$

Because the last term $c\bar{N}$ lives entirely in the tangent plane to the surface $\hat{\sigma}$ at P, it has a decomposition relative to the basis $\{\hat{\sigma}_1, \hat{\sigma}_2\}$,

$$c\bar{N} = \hat{a}\hat{\sigma}_1 + \hat{b}\hat{\sigma}_2,$$

Thus, we have

$$\delta U = \hat{U} - U = a\sigma_1 + b\sigma_2 + (-\hat{a}\hat{\sigma}_1 - \hat{b}\hat{\sigma}_2),$$

or,

$$\hat{U} + (\hat{a}\hat{\sigma}_1 + \hat{b}\hat{\sigma}_2) = U + a\sigma_1 + b\sigma_2$$

Consequently, we have the **evolution vector field** with two equivalent representations (over two different basis of $\mathbb{R}^3$),

$$\eta = U + a\sigma_1 + b\sigma_2 \tag{2}$$

$$\hat{\eta} = \hat{U} + \hat{a}\hat{\sigma}_1 + \hat{b}\hat{\sigma}_2 \tag{3}$$

This vector field is defined on a neighborhood of the point P in $\mathbb{R}^3$, rather than just on the surfaces.

Next, for any point P which lies on a curve of intersection for the deforming surfaces, we can define a scalar field ϕ in a neighborhood of P (in $\mathbb{R}^3$). By the inverse function theorem, there is a neighborhood of P which is entirely covered by each deforming family. For a point P' in this neighborhood, we define $\phi(P') = \hat{t} - t$, where $\hat{t}$ (resp. t) is the time when the surface $\hat{\sigma}(\hat{s}, t)$ (resp. $\sigma(s, t)$) reaches P'. Although ϕ is not defined everywhere on $\mathbb{R}^3$, it is defined on a neighborhood of $\mathcal{L}_0$.

The following properties involving ϕ, η, and $\mathcal{L}_0$ can be shown to hold:

1. The directional derivative $\frac{\partial \phi}{\partial \eta} = 0$ identically wherever ϕ is defined.
2. The zero level set of ϕ is exactly $\mathcal{L}_0$.
3. Hence, η is tangent to $\mathcal{L}_0$ at all points.

Now suppose point P is on $\mathcal{L}_0$, and lies on an intersection curve at time t. The condition that η is tangent to $\mathcal{L}_0$ allows us to follow the evolution of P on future intersection curves by solving the differential equation

$$\frac{dx}{dt} = \eta(x) \quad \text{with initial condition } x(0) = P$$

for $x(t) \in \mathbb{R}^3$. The evolution vector field η is the image of the vector field $\xi = \frac{\partial}{\partial t} + a\frac{\partial}{\partial s_1} + b\frac{\partial}{\partial s_2}$ under the parametrization map σ. Thus, the evolution could likewise be followed on the parameter space using instead the vector field ξ, and analogously for $\hat{\sigma}$.

Then, we can use a discrete algorithm for solving the differential equations to follow the evolution of the intersection curves over a time interval void of transitions. Specifically, for small time dt P moves to $Q = P + dt\ (a\sigma_1 + b\sigma_2)$ on the physical surface and if $p = s \in \mathbb{R}^2$ corresponds to P then $q = s + (a\ dt, b\ dt)$ will correspond to Q in the parameter space, and analogously for $\hat{\sigma}$. The first order marching algorithm accumulates error over time, so point correction can be used to increase the quality. Various point correction algorithms are discussed in [**?**] in the context of static surface-surface intersection. We have adopted the middle point algorithm as presented in [**?**] to relax the points onto the actual intersection curve.

Also notice that we are not tracing out an entire intersection loop from some starting point; instead, we represent intersection curves by an ordered list of sample points. Sample points are adaptively inserted or deleted so that the spacing of two consecutive sample points is neither too far away nor too close, and so that the angle deviation of 3 consecutive sample points stays small.

3 Formulation in the Augmented Parametric Space

Define a vector distance mapping

$$d(s, \hat{s}, t) = \hat{\sigma} - \sigma : \mathbb{R}^5_{\{s,\hat{s},t\}} \longrightarrow \mathbb{R}^3 \tag{4}$$

where $\mathbb{R}^5_{\{s,\hat{s},t\}}$ [3] is the combined parametric space of the 2 surfaces and the time domain, and is thus called **augmented parametric space**. The canonical orthonormal basis $\mathbb{R}^5_{\{s,\hat{s},t\}}$ is denoted as $\{e_{s_1}, e_{s_2}, e_{\hat{s}_1}, e_{\hat{s}_2}, e_t\}$. The 0-set of this mapping, denoted $\mathcal{I}$ hereafter in this paper, gives the set of all intersection points in $\mathbb{R}^5_{\{s,\hat{s},t\}}$. Note that $d(s,\hat{s},t)$ concisely represents related equations for three separate coordinate functions. Considering the x-component $d_x(s,\hat{s},t)$. $d_x(s,\hat{s},t) = 0$ defines a hyper-surface in $\mathbb{R}^5_{\{s,\hat{s},t\}}$, with corresponding normal

$$N_x = \nabla d_x = (-x_1, \ -x_2, \ \hat{x}_1, \ \hat{x}_2, \ \delta U_x) \tag{5}$$

The component functions y and z define another two hyper-surfaces with analogous expressions for their normals N_y and N_z. Geometrically, $\mathcal{I}$ is the locus of intersection points of these three hyper-surfaces in $\mathbb{R}^5_{\{s,\hat{s},t\}}$. The Jacobian [**?**] of the mapping $d(s,\hat{s},t) : \mathbb{R}^5_{\{s,\hat{s},t\}} \longrightarrow \mathbb{R}^3$ is,

$$\mathcal{J} = (N_x \ N_y \ N_z)^t = \begin{pmatrix} -x_1 & -x_2 & \hat{x}_1 & \hat{x}_2 & \delta U_x \\ -y_1 & -y_2 & \hat{y}_1 & \hat{y}_2 & \delta U_y \\ -z_1 & -z_2 & \hat{z}_1 & \hat{z}_2 & \delta U_z \end{pmatrix} = (-\sigma_1 \quad -\sigma_2 \quad \hat{\sigma}_1 \quad \hat{\sigma}_2 \quad \delta U). \tag{6}$$

Remark 1 If the 2 tangent planes to the two deforming surfaces at the intersection point are not the same, then both of the triple scalar products (determinants) $[\sigma_1\sigma_2\hat{\sigma}_i]$'s $(i = 1,2)$ can not simultaneously vanish, and so $\mathcal{J}$ has the full rank of 3. Otherwise, the two tangent planes must be the same. Assuming, at such a touching point, δU is not on the common tangent plane, i.e., $[\hat{\sigma}_1\hat{\sigma}_2\delta U] \neq 0$ and $[\sigma_1\sigma_2\delta U] \neq 0$, $\mathcal{J}$ again has the full rank. Therefore, the 0-set of the distance mapping $d(s,\hat{s},t) = \hat{\sigma} - \sigma : \mathbb{R}^5_{\{s,\hat{s},t\}} \longrightarrow \mathbb{R}^3$, is a well defined implicit 2-manifold in the augmented parametric space.

4 Transition of Intersection Loops

In singularity theory, the situation we consider is considered generic. That is, except for a finite set of times, the two closed surfaces intersect transversely, that is, at each intersection point the tangent planes of the surfaces are different. Thus, the method presented in Section 2 can be applied to track the evolution of the curves. Over such time intervals topological changes are guaranteed not to occur.

At the remaining finite number of times, there will be intersection points at which the tangent planes coincide (non-transverse points). Again for *generic deformations*, singularity theory describes exactly the transitions in intersection curves that can occur as the evolution passes such times. These transitions can always be given (up to a change of coordinates) by standard model equations, so there is essentially a unique way for each transition to occur. We shall refer to points (and times) at which transitions occur as *transition points*. These transitions are classified as,

[3] $\mathbb{R}^5_{\{s,\hat{s},t\}}$ denotes $\mathbb{R}^5$ with the five coordinates being $s_1, s_2, \hat{s}_1, \hat{s}_2, t$ and analogously, for $\mathbb{R}^3_{\{s,t\}}$, etc.

1. *a creation event*, when a new intersection loop is created (Fig. 2),
2. *an annihilation event*, when one of the current loops collapses and disappears (Fig. 2 in the reverse direction),
3. *an exchange event*, when two branches of intersection curves meet and exchange branches (Fig. 3).

The exchange event can have two different global consequences. If the two branches are part of the same curve, an intersection loop is split into 2 loops and we refer to this as a *splitting event* (Fig. 6). If the branches are from distinct intersection loops, a single loop is formed in a *merge event* (Fig. 6 in reverse order).

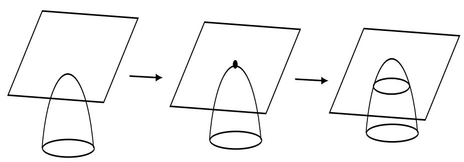


Fig. 2. Creation of IC Component

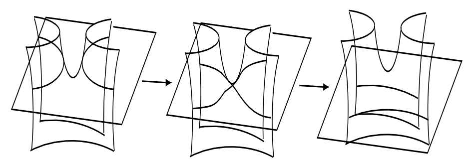


Fig. 3. Exchange of IC Components

4.1 Detection of Transition Events

In this sub-section, we formulate the topological transition points as the 0-set of a rational system of 5 nonlinear constraints in 5 variables. The 0-set has dimension 0, i.e., it is a discrete collection of points. It can be *robustly and efficiently* computed using a rational constraint solver [**?**, **?**]. The robustness is achieved by bounding the subdivided implicit surface $\mathcal{I}$ with the corresponding hyper-tangent cone [**?**], an extension of the bounding tangent cones for explicit plane curves and explicit surfaces [**?**, **?**].

Let us recall that the implicit 2-manifold $\mathcal{I}$ in $\mathbb{R}^5_{\{s,\hat{s},t\}}$ is the locus of intersection points of the two deforming surfaces, over the whole time period. Geometrically, the intersection curves, at some time point, are the corresponding height contour of $\mathcal{I}$ when the t is regarded as the vertical axis. Therefore, it is obvious that there will be one of the three transition events listed earlier, if the tangent space to $\mathcal{I}$ at a point $(s, \hat{s}, t)$ is orthogonal to the t-axis. Since $\mathcal{I}$ and its tangent space have the same dimension, namely, 2, the orthogonality condition is tantamount to satisfying two equations,

$$T_1 \cdot e_t = 0, \qquad T_2 \cdot e_t = 0,$$

where T_1 and T_2 are any two vectors spanning the tangent space. A simple and natural way to construct such a pair of tangent vectors is to let T_1 be the tangent to an s_2-iso-curve on $\mathcal{I}$ with the extra constraint $s_2 = c_2$ for some constant c_2, and let T_2 be the tangent to an s_1-iso-curve on $\mathcal{I}$ with the extra constant $s_1 = c_1$

for some constant c_1. Noticing that an s_2-iso-curve is the intersection of 4 hyper surfaces in $\mathbb{R}^5_{\{s,\hat{s},t\}}$, defined by $s_2 = c_2$, $d_x = 0$, $d_y = 0$, and $d_z = 0$,

$$T_1 = \begin{vmatrix} e_{s_1} & e_{s_2} & e_{\hat{s}_1} & e_{\hat{s}_2} & e_t \\ 0 & 1 & 0 & 0 & 0 \\ -x_1 & -x_2 & \hat{x}_1 & \hat{x}_2 & \delta U_x \\ -y_1 & -y_2 & \hat{y}_1 & \hat{y}_2 & \delta U_y \\ -z_1 & -z_2 & \hat{z}_1 & \hat{z}_2 & \delta U_z \end{vmatrix} = (\mathscr{T}^{\hat{1}\hat{2}\delta},\ 0,\ \ \mathscr{T}^{1\hat{2}\delta},\ \ -\mathscr{T}^{1\hat{1}\delta},\ \ \mathscr{T}^{1\hat{1}\hat{2}}),$$

where $\mathscr{T}$' denotes the triple scalar product of its 3 corresponding vectors indicated by the superscripts. Superscripts i and $\hat{i}$ represent σ_i and $\hat{\sigma}_i$, respectively, while a superscript δ represents δU (e.g., $\mathscr{T}^{\hat{1}\hat{2}\delta} = [\hat{\sigma}_1\ \hat{\sigma}_2\ \delta U]$). A similar derivation exists for $\mathscr{T}_2$, and in general, we have,

$$T_i = \mathscr{T}^{\hat{1}\hat{2}\delta}\ e_{s_i} + \mathscr{T}^{i\hat{2}\delta}\ e_{\hat{s}_1} - \mathscr{T}^{i\hat{1}\delta}\ e_{\hat{s}_2} + \mathscr{T}^{i\hat{1}\hat{2}}\ e_t, \quad i = 1, 2. \tag{7}$$

At transition points, the last component of T_1 and T_2 vanishes, i.e.

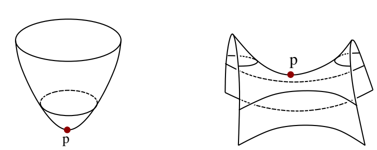
$$T_1 \cdot e_t = \mathscr{T}^{1\hat{1}\hat{2}} = [\sigma_1\ \hat{\sigma}_1\ \hat{\sigma}_2] = 0, \quad T_2 \cdot e_t = \mathscr{T}^{2\hat{1}\hat{2}} = [\sigma_2\ \hat{\sigma}_1\ \hat{\sigma}_2] = 0, \tag{8}$$

Remark 2 By Remark 1, $\mathscr{T}^{\hat{1}\hat{2}\delta} \neq 0$ at any transition point. Therefore, at a transition point, T_1 and T_2 are guaranteed to be independent of each other. It is also easily seen that Eq. (8) simply require the two tangents σ_1 and σ_2 to the first offset surface to be perpendicular to the normal of the second offset surface, i.e., the two tangent planes to the two deforming surfaces in the euclidean space are coincident.

Finally, together with $\hat{\sigma} - \sigma = 0$, Eq. (8) gives a rational system with 5 constraints in 5 variables, whose 0-dimensional solution set contains all the transition points we are seek.

4.2 Compute the Structural Change at Transition Events

In this section, we perform the shape computation of the 2-manifold $\mathcal{I}$ at a transition point, and subsequently compute the corresponding transition event by contouring the osculating paraboloid [**?**, **?**] to the local shape (Fig. 4).



(a) elliptic (b) hyperbolic

Fig. 4. Contour Osculating Paraboloid

The implicit surface $\mathcal{I}$ is a 2-manifold in a 5-space $\mathbb{R}^5_{\{s,\hat{s},t\}}$. Shape computation is difficult because it is an implicit surface, and also because its codimension $\neq 1$.

Most recently, a comprehensive set of formulas for curvature computation on implicit curves/surfaces with further references were presented in [**?**]. However, it is limited to curves/surfaces embedded in $2D$ or $3D$ spaces. There exists some literature from the visualization community, e.g., [?, ?] and references therein, that develops second order derivative computation on iso-surfaces extracted from trivariate functions. Most of these approaches use discrete approximations. Recently, [?] developed B-spline representations for the Gaussian curvature and squared mean curvature of the iso-surfaces extracted from volumetric data defined as a trivariate B-spline function, and subsequently presented an exact curvature computation for every possible point of the 3D domain. While we seek an exact differential computation, the task here significantly differs from that in [?] and [?, ?] since the implicit 2-manifold $\mathcal{I}$ has codimension 3. In [?], a set of formulas for computing Riemannian curvature, mean curvature vector, and principal curvatures, specifically for a 2-manifold, and with *arbitrary* codimension, is presented. The specific 2nd order problem we seek to solve, namely initializing the newly created intersection loop, or switching the two pairs of hyperbolic-like segments, is based on *shape approximation*. For a surface in 3-space, the local shape approximation is simply the osculating quadric, expressed in the second fundamental form [?] as $z = II(a, a)$ where a is any tangent vector, and z is the vertical distance from the local surface point to the tangent plane. Observing that the second order shape approximation is best done if the codimension is 1, we do not compute the second fundamental form directly on the 2-manifold in 5-space. Instead, we project the 2-manifold to a 3-space of either $\mathbb{R}^3_{\{s,t\}}$ (our choice in this paper)or $\mathbb{R}^3_{\{\hat{s},t\}}$. The second fundamental form is then computed for this projected 2-manifold, and shape approximation is achieved subsequently. Notice that the shape approximation in the projected 3-space gives only a partial answer to the transition event; the full solution is achieved by the tangential mapping between the projected 2-manifold and the original one (cf. Observation 1 below).

Projection of $\mathcal{I}$ to $\mathbb{R}^3_{\{s,t\}}$ Near a critical point, T_1 and T_2 (cf. Eq. (7)) give two vectors spanning the tangent space to $\mathcal{I}$. By projecting $\mathcal{I}$ onto $R^3_{\{s_1,s_2,t\}}$ and ignoring the $\hat{s}_1$ and $\hat{s}_2$ components, we transform $\mathcal{I}$, a 2-manifold in $\mathbb{R}^5_{\{s,\hat{s},t\}}$, into a surface in $R^3_{\{s,t\}}$, denoted as $\mathcal{I}^s$. Furthermore, the projection, denoted as π hereafter, is a diffeomorphism. and the projected tangent plane (the tangent plane to $\mathcal{I}^s$) is spanned by,

$$T_1^s = \mathscr{T}^{\hat{1}\hat{2}\delta}\ e_{s_1} + \mathscr{T}^{1\hat{1}\hat{2}}\ e_t, \quad T_2^s = \mathscr{T}^{\hat{1}\hat{2}\delta}\ e_{s_2} + \mathscr{T}^{2\hat{1}\hat{2}}\ e_t, \tag{9}$$

where we have used the superscript s to distinguish the tangents from their counterparts of $\mathcal{I}$ in the original augmented parametric space $\mathbb{R}^5_{\{s,\hat{s},t\}}$.

Exactly at the transition point where $\mathscr{T}^{1\hat{1}\hat{2}} = \mathscr{T}^{2\hat{1}\hat{2}} = 0$ (cf. Eq. (8)), we have,

$$T_1^s = \mathscr{T}^{\hat{1}\hat{2}\delta}\ e_{s_1}, \quad T_2^s = \mathscr{T}^{\hat{1}\hat{2}\delta}\ e_{s_2}.$$

Hereafter, a point in the tangent space $\mathbf{TS}_{\mathcal{I}^s}$ is typically specified by its 2 coordinates, say, a_1 and a_2, with respect to the basis $\{T_1^s, T_2^s\}$, the canonical frame $\{e_{s_1}, e_{s_2}\}$ scaled by $\mathscr{T}^{\hat{1}\hat{2}\delta}$.

Observation 1 *At a transition point, the inverse of the tangent map of* π *is (cf. Eqs.* (7)*),*

$$(1,0) \mapsto (\mathscr{T}^{\hat{1}\hat{2}\delta}, 0, \mathscr{T}^{1\hat{2}\delta}, -\mathscr{T}^{1\hat{1}\delta}, 0), \qquad (0,1) \mapsto (0, \mathscr{T}^{\hat{1}\hat{2}\delta}, \mathscr{T}^{2\hat{2}\delta}, -\mathscr{T}^{2\hat{1}\delta}, 0),$$

where $(1,0)$ *and* $(0,1)$ *are the coordinates of two points in the local tangent space* $\mathbf{TS}_{\mathcal{I}^s}$ *with basis* $\{T_1^s, T_2^s\}$*.*

The Shape Computation The local shape of $\mathcal{I}^s$ in $\mathbb{R}^3_{\{s,t\}}$ is determined from the second fundamental form II. At *a transition point*,

$$II = \left(N^s \cdot \nabla_{T_i^s} T_j^s\right) = \left(\nabla_{T_i^s} T_j^s \cdot e_t\right), \tag{10}$$

and the local shape is approximated by the osculating quadric [**?**, **?**],

$$\delta t = II(a,a) = \begin{pmatrix} a_1 & a_2 \end{pmatrix} II \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{11}$$

where $a = (a_1, a_2) \in \mathbf{TS}_{\mathcal{I}^s}$. Notice that we wrote the left hand side as δt, because, at a transition point, the tangent plane $\mathbf{TS}_{\mathcal{I}^s}$ is horizontal, and thus the local vertical height is exactly the time deviation from the considered transition point.

The covariant derivatives are best computed in the original 5-space, i.e.,

$$\nabla_{T_i^s} T_j^s \cdot e_t = \nabla_{T_i} T_j \cdot e_t.$$

By Eq. (7),

$$\begin{aligned}\nabla_{T_i^s} T_j^s \cdot e_t &= \nabla_{T_i} T_j \cdot e_t = \nabla_{T_i}(T_j \cdot e_t) = \nabla_{T_i} \mathscr{T}^{j\hat{1}\hat{2}} \\ &= \mathscr{T}^{\hat{1}\hat{2}\delta}\ \frac{\partial \mathscr{T}^{j\hat{1}\hat{2}}}{\partial s_i} + \mathscr{T}^{i\hat{2}\delta}\ \frac{\partial \mathscr{T}^{j\hat{1}\hat{2}}}{\partial \hat{s}_1} - \mathscr{T}^{i\hat{1}\delta}\ \frac{\partial \mathscr{T}^{j\hat{1}\hat{2}}}{\partial \hat{s}_2}.\end{aligned}$$

Introducing the following notations ($i, j, k \in \{1, 2\}$),

$$\mathscr{T}^{j_i\hat{1}\hat{2}} = [\frac{\partial \sigma_j}{\partial s_i}\ \ \hat{\sigma}_1\ \ \hat{\sigma}_2], \quad \mathscr{T}^{i\hat{1}_k\hat{2}} = [\sigma_i\ \ \frac{\partial \hat{\sigma}_1}{\partial \hat{s}_k}\ \ \hat{\sigma}_2], \quad \mathscr{T}^{i\hat{1}\hat{2}_k} = [\sigma_i\ \ \hat{\sigma}_1\ \ \frac{\hat{\sigma}_2}{\partial \hat{s}_k}]$$

yields,

$$\nabla_{T_i^s} T_j^s \cdot e_t = \mathscr{T}^{\hat{1}\hat{2}\delta}\ \mathscr{T}^{j_i\hat{1}\hat{2}} + \mathscr{T}^{i\hat{2}\delta}\ (\mathscr{T}^{j\hat{1}_1\hat{2}} + \mathscr{T}^{j\hat{1}\hat{2}_1}) - \mathscr{T}^{i\hat{1}\delta}\ (\mathscr{T}^{j\hat{1}_2\hat{2}} + \mathscr{T}^{j\hat{1}\hat{2}_2}).$$

Throughout this paper, we make the generic assumption that the transition point is non-degenerate, i.e., $det(II) \neq 0$.

Heuristically Uniform Sampling of Local Height Contours To compute various transition events, the height contour curves of the local osculating quadric needs to be uniformly sampled in the *euclidean space* $\mathbb{R}^3$.

Suppose we are sampling the height contour with the time deviation δt. By Eq. (11), the sample point $p_v \in \mathbf{TS}_{\mathcal{I}^s}$ along a direction $v \in \mathbf{TS}_{\mathcal{I}^s}$, is $p_v = \sqrt{\frac{2\delta t}{II(v,v)}}\ v$. Therefore, given an initial list of sample *directions*, the following algorithm generates a list of heuristically uniform sample points.

Algorithm 1 Heuristically Uniform Sampling

1. *Turn the given list of sample* directions *into a list of sample* points *by scaling each element* p *by* $\sqrt{\frac{2\delta t}{II(p,p)}}$.
2. *In the current list, find a neighboring sample pair* $p, q \in \mathbf{TS}_{\mathcal{I}^s}$ *with maximal distance.*
3. *Let* $m = \frac{p+q}{2}$*, scale* m *by* $\sqrt{\frac{2\delta t}{II(m,m)}}$*, and insert it into the list in between* p *and* q.
4. *If not enough sample points, or the distances are not approximately uniform, goto* **Step 2**.

Compute Transition Events

Compute Creation Events: If $det(II) > 0$ (or equivalently, the Gaussian curvature of $\mathcal{I}^s$ is positive), the osculating quadric (Eq. (11)) is an elliptic paraboloid, and the transition point has elliptic type. See Fig. 4(a).

For an upward elliptic type and offset surfaces deforming forward, or for a downward elliptic type and offset surfaces deforming backward, a creation event is occurring, i.e., an entirely new intersection loop is created from nothing. The following algorithm computes the intersection loop at the time deviation δt from the transition point.

Algorithm 2 Compute Ellipse Contour for a Creation Event

1. *Put directions* $(1,0), (0,1), (-1,0)$ *into the ordered list of directions* $\mathcal{V}$.
2. *Apply Algo. 1 to transform* $\mathcal{V}$ *to a ordered list of uniform samples in* $\mathbf{TS}_{\mathcal{I}^s}$.
3. *Except the first and the last ones, copy and negate in order all elements, in* $\mathcal{V}$*, and append to itself.*
4. *Map* $\mathcal{V}$ *to a ordered list of samples in* $\mathbb{R}^4_{\{s,\hat{s},t\}}$ *(cf. Observation 1).*

Compute Annihilation Events: At an upward elliptic transition point when offset surfaces deform backward (in time), or at a downward elliptic transition point when offset surfaces deforming forward, there is an *annihilation event* happening, i.e. an intersection loop collapses and disappears. See Fig. 4(a). The key issue here is to choose the right current intersection loop to annihilate. If there is currently only one intersection loop, annihilate it. Otherwise, we use an "evolve-to-annihilate" strategy as illustrated in Fig. 5. First, evolve all intersection loops at time t_1 to the time t' (i.e., the contour position used for the pre-computation of the corresponding creation event). Then, using the inclusion test [**?**], find the one that the critical point p identifies to annihilate.

Compute Switch Events: If $det(II) < 0$ (or equivalently, the Gaussian curvature of $\mathcal{I}^s$ is negative), the osculating quadric (Eq. (11)) is a hyperbolic paraboloid, and the transition point has hyperbolic type. See Fig. 4(b).

Deforming across a hyperbolic transition point is a quite different situation from an elliptic point inasmuch as there is a switch of two pairs of hyperbolic-like

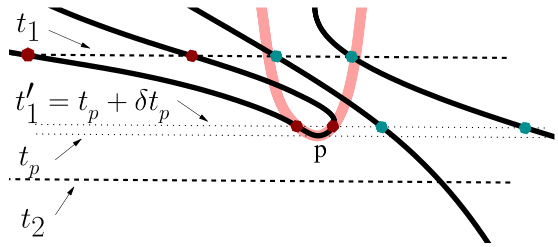


Fig. 5. Evolve to Annihilate

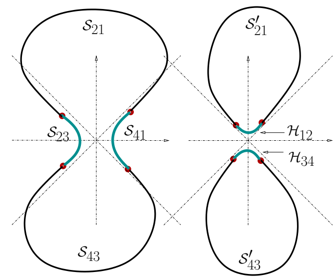


Fig. 6. Split/Merge

segments (cf. Fig. 3 in the euclidean space, and Fig. 4 of $\mathcal{I}^s$): 2 local segments approach each other, say, from *above* the transition point p , touch *at* p, and then swap and depart into another two local segments *below* p. If the approaching pair of segments is from one intersection loop (Fig. 6), we have a split event; and if it is from 2 intersection loops (Fig. 6 in reverse order), we have a merge event. Each segment is a height contour of the local shape approximated by the osculating hyperbolic paraboloid. The following algorithm computes one pair of such height contours.

Algorithm 3 Compute Hyperbolic Contours for a Switch Event

1. *Put directions* $u_1 + (u_2 - u_1) * \lambda, u_1 + u_2, u_2 + (u_1 - u_2) * \lambda$ *into the ordered list of directions* $\mathcal{V}$*.*
2. *Invoke Algo. 1 to transform* $\mathcal{V}$ *to an ordered list of uniform samples in* $\mathbf{TS}_{\mathcal{I}^s}$*.*
3. *In order, copy and negate all elements into another list* $\mathcal{V}'$*.*
4. *Map* $\mathcal{V}$ *to an ordered list of samples in* $\mathbb{R}^4_{\{s,\hat{s},t\}}$ *(cf. Observation 1). Do the same for* $\mathcal{V}'$*.*

In the algorithm, u_1 and u_2 are the two asymptotic directions, which can be solved (for u) from the equation $II(u,u) = 0$ using the second fundamental form in Eq. (10). The other pair of contours, with the opposite height value, can be sampled similarly, with one of the asymptotic directions reversed.

Based on the deforming direction we can determine which of the 2 principal curvature directions is the approaching direction, and which is the departing direction. Then the approaching pair of segments of current intersection curves is the one that is closest to the considered transition point along the approaching direction. Using Algorithm 3, the switch event can be computed by cutting the two approach segments, evolving the rest across, say upward, p, and then pasting the other pair of contours to the departing pair of segments (Illustrated Fig. 6). Finally, Fig. 7 gives an example of split event of two deforming torus-like surfaces. For demo videos, see http://www.cs.utah.edu/∼xchen/papers/more.html

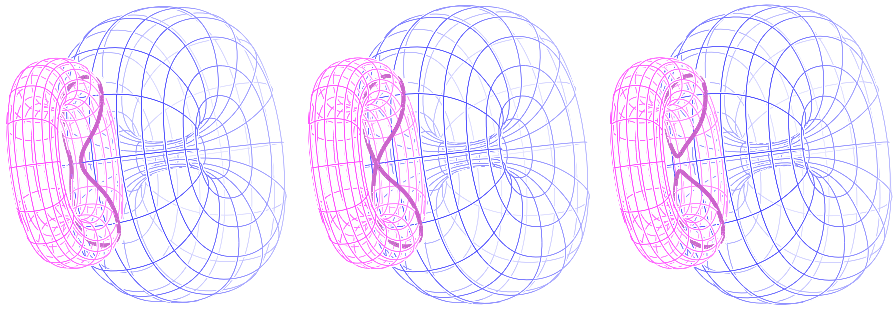


Fig. 7. Transition: A split event of 2 deforming torus-like surfaces

5 Conclusion

In this paper, we have applied a mathematical framework provided by singularity theory to develop algorithms for continuously and robustly tracking the intersection curves of two generically deforming surfaces, on the assumption that both the base surfaces and the deforming vectors have rational parametrization. The core idea is to divide the process into two steps depending on when transition points occur. Away from any transition points, the intersection curves evolve without any structural change. We found a simple and robust method which constructs an evolution vector field directly in the euclidean space R^3 and evolves the intersection curves accordingly.

We further developed a method for identifying transition points and following topological changes in the intersection curves through the introduction of an implicit 2-manifold $\mathcal{I}$, which consists of the union of intersection curves in the augmented (by time domain t) joint parameter space. The transition points are identified as the points on $\mathcal{I}$ where the tangent spaces are orthogonal to t-axis, and the topological change of the intersection curves is subsequently computed by 2nd order differential geometric computations on $\mathcal{I}$.

There are further transitions which can occur for deforming surfaces, including the surface developing singularities, self-intersections, and triple intersection points.. We are now developing a similar formulation for tracking the intersection curve end points that correspond to surface boundaries, and for tracking triple intersection points.